A definition and spectral properties of self-adjoint operators derived from the Schrödinger operator with the white noise potential on the plane

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https://www.math.h.kyoto-u.ac.jp/users/ueki/2DWN-Loc1.pdf

https://www.math.h.kyoto-u.ac.jp/users/ueki/presen20250901-5FJCPI

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Motivation

Study the spectral property of the Schrödinger operator

$$-\Delta + \xi \tag{1}$$

in the case that the potential is the whitenoise on $\mathbb{R}^d$

$\xi = (\xi(x))_{x \in \mathbb{R}^d}$: a system of the i.i.d. Gaussian random variables

Gaussian random field, $\mathbb{E}[\xi(x)] = 0$, $\mathbb{E}[\xi(x)\xi(y)] = \delta(x-y)$

Difficulty: "$x \mapsto \xi(x)$"$\in C^{-\epsilon-d/2}$, not a regular function

Dimensions

$$d = 1 \Rightarrow W^{1,2}([a,b]) \ni f, g \mapsto \int_a^b (f'g' + \underset{C_{loc}^{-1/2-\epsilon}}{\xi} fg)dx \ : \text{ well-defined}$$

$-\Delta + \xi$ is realized as a self-adjoint operator

M. Fukushima and S. Nakao (1977) Spectral asymptotics

(Asymptotics of the Integrated density of states)

N. Minami (1988) (1989) Exponential Localization at all energies

(Generarization: $\xi \to \partial$(Lévy process))

L. Dumaz and C. Labbé (2020) (2023) Eigenvalues Eigenvectors Statistics

Recently

$d = 2$ or $3 \Rightarrow -\Delta + \lim_{\varepsilon \to 0}(\xi_\varepsilon(x) + c_\varepsilon)$ are realized as self-adjoint operators

$d \geq 4 \Rightarrow$ No results

Related works on singular SPDEs

M. Hairer (2014) The theory of regularity structures,
M. Gubinelli, P. Imkeller and N. Perkowski (2015) Paracontrolled calculus
A. Kupiainen (2016) Renormalization Group
$\Rightarrow$ Stochastic quatization equation for ϕ^4_3 Euclidean quantum field theory
Generalized continuous parabolic Anderson model
Kardar–Parisi–Zhang type equation
Navier-Stokes equation with very singular forcing
and so on

Eg. Continuous parabolic Anderson model

$$\partial_t u(t,x) = \Delta_x u(t,x) - \lim_{\varepsilon\to 0}(\xi_\varepsilon(x) + c_\varepsilon) u(t,x) \text{ for } t > 0$$

Schrödinger operators on compact spaces

R. Allez and K. Chouk (2015) Paracontrolled calculus based on Fourier Analysis

$-\Delta + \lim_{\varepsilon\to 0}(\xi_\varepsilon(x) + c_\varepsilon)$ on $\mathbb{R}^2/\mathbb{Z}^2$: Self-adjoint, Norm resolvent limit of C^∞-app.

↑Fourier partial sum

(Discrete Spectrum, Asymptotic Distribution)

M. Gubinelli, B. Ugurcan and I. Zachhuber (2020) Extension to $\mathbb{R}^3/\mathbb{Z}^3$

$$i\partial_t u(t,x) = \underbrace{(\Delta - \lim_{\varepsilon\to 0}(\xi_\varepsilon(x) + c_\varepsilon) - k_\xi)}_{\geq 0} u(t,x) - (u|u|^2)(t,x), \underset{\text{given}}{u(0,\cdot)}$$

$$\partial_t^2 u(t,x) = (\Delta - \lim_{\varepsilon\to 0}(\xi_\varepsilon(x) + c_\varepsilon) - k_\xi) u(t,x) - (u^3)(t,x), \underset{\text{given}}{(u(0,\cdot), \partial_t u(0,\cdot))}$$

Well-posedness,

The convergence of the solutions of regularized equations

C. Labbé (2019) similar results by the theory of regularity structure for

$-\Delta + \lim_{\varepsilon\to 0}(\xi_\varepsilon(x) + c_\varepsilon)$ on $(-L, L)^{2or3}$ with periodic or Dirichlet conditions

↑convolution with C_0^∞ function

Extensions to noncompact spaces

M. Hairer and C. Labbé, (2015) (2018)

$$\partial_t u = \Delta u - \lim_{\varepsilon \to 0} (\xi_\varepsilon + c_\varepsilon) u, \; t > 0, x \in \mathbb{R}^d, \; d = 2, 3, \; u(0, \cdot)$$

↑convolution with C_0^∞ function — given

Well-posedness,

The convergence of the solutions of regularized equations

Y. Hsu and C. Labbé, (2024)

Construct a self adjoint operator $-\Delta + \lim_{\varepsilon \to 0} (\xi_\varepsilon + c_\varepsilon)$ on $\mathbb{R}^d$, $d = 2, 3$,

↑convolution with C_0^∞ function

as the generator of the parabolic Anderson model

For the operator, Spec$= \mathbb{R}$

Related operators on noncompact spaces

B. Ugurcan, (2022) $-\Delta + \lim_{\varepsilon\to 0}(\xi_\varepsilon(x) + \widetilde{c}_\varepsilon(x))$ on $\mathbb{R}^2$ with $\widetilde{c}_\varepsilon(x) \xrightarrow{|x|\to\infty} 0$

$$= \underline{-\Delta + \lim_{\varepsilon\to 0}(\xi^{\uparrow}_\varepsilon(x) + \widetilde{c}_\varepsilon(x))} + \underline{\xi^{\downarrow}_\varepsilon(x)},$$ extension of GUZ(2020)

$\uparrow$ An extension of the method for the compact case

$\nwarrow$ By a commutator estimate,

where $\xi = \xi^{\uparrow}(x) + \xi^{\downarrow}(x)$ (Smooth functions in x)

$$= \sum_{n=-1}^{\infty} \widetilde{\chi_{[[2^n],2^{n+1}]}}(|x|)\{\underbrace{\widetilde{\chi_{[c2^n,\infty)}}(-\Delta)\xi}_{\text{high energy pert}} + \underbrace{\widetilde{\chi_{[0,c2^n]}}(-\Delta)\xi}_{\text{low energy pert}}\}$$

$\widetilde{c}_\varepsilon(x) = \mathbb{E}[\text{A resonant product of } \xi^{\uparrow}_\varepsilon(x) \text{ and } (1-\Delta)^{-1}\xi^{\uparrow}_\varepsilon(x)]$

Heat semigroup approach in the paracontrolled calculus

Use the heat semigroup to multiply functions

(cf. the more traditional approach uses Fourier analysis.)

(Applicable to many kind of configuration spaces as manifolds, graphs,....)

I. Bailleul and F. Bernicot (2016) For generalized PAM on $\underset{\text{without compactness}}{\text{2D manifold}}$

Well-posedness, The convergence of the solutions of regularized equations

I. Bailleul, F. Bernicot and D. Frey (2018)

For PAM and multiplicative Burgers eq. on $\underset{\text{without compactness}}{\text{3D manifold}}$

Well-posedness, The convergence of the solutions of regularized equations

A. Mouzard (2022) Self-adjointness, Norm-resolvent limit of C^∞-app. for

$$\underset{\text{eg. Lap.Belt}}{-\Delta+} \lim_{\varepsilon\to 0}\big(\underset{\text{e.g. } (e^{\varepsilon^2\Delta}\xi)(x)}{\xi_\varepsilon(x)} + c_\varepsilon(x)\big) \text{ on 2D compact manifold.}$$

$$c_\varepsilon(x) \equiv c_\varepsilon \text{ on } \mathbb{R}^2/\mathbb{Z}^2$$

Our Topics

By the paracontrolled calcuclus by the heat semigroup referring Mouzard (2020) and the partition of unity,

$$\widetilde{H^\xi} = -\Delta + \lim_{\varepsilon\to 0}\Big(\underset{(e^{\varepsilon^2\Delta}\xi)(x)}{\underset{||}{\xi_\varepsilon(x)}} + c_\varepsilon\Big) \text{ on } \mathbb{R}^2.$$

Self-adjointness.
$\mathrm{Spec}(\widetilde{H^\xi}) = \mathbb{R}$ as in Hsu and Labbé (2024).
Anderson localization at sufficiently low energies by a traditional proof.
The locaization is a central topic relating to random operators.
The localization is a phenomenon caused by the randomness of the environment.
(The operator should be stationary.)

Comparison between Hsu-Labbe's definition and our definition

Advantages of Hsu-Labbe's definition

▷ The 3-dimensional case is included.

▷ The expressions are simpler.

▷ $\xi_\varepsilon(x) = \xi * \rho(\cdot/\varepsilon)/\varepsilon^d$ with a general C_0^∞ function ρ.

(cf. $\xi_\varepsilon(x) = (e^{\varepsilon^2 \Delta}\xi)(x)$ basically in the heat semigroup approach.)

Advantages of our definition

▷ More convenient for arguments that handle the operator directly.

Examples: ◦ Our proof of "Spec$(\widetilde{H^\xi}) = \mathbb{R}$" approaches $\widetilde{H^\xi}$ directly.

◦ The most used proof of the Anderson localization is a combination of several results, each of which is valuable in its own right.

The proof may be better suited to our definition.

Products $fg = P_f g + \Pi(f,g) + P_g f + P_1^{(b)}((P_1^{(b)} f)(P_1^{(b)} g))$

$0 << b \in 2\mathbb{Z}$ fixed $\qquad P_t^{(b)} = \sum_{j=0}^{b-1} \frac{(-t\Delta)^j}{j!} e^{t\Delta}$

$$\underbrace{P_0^{(b)}((P_0^{(b)} f)(P_0^{(b)} g))}_{=fg} - P_1^{(b)}((P_1^{(b)} f)(P_1^{(b)} g)) = -\int_0^1 dt\partial_t \{P_t^{(b)}((P_t^{(b)} f)(P_t^{(b)} g))\}$$

$$= P_f g + \Pi(f,g) + P_g f$$

$P_f g := \sum_\nu c_\nu \int_0^1 \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g))$: paraproduct(a well-defind distribution)

$\Pi(f,g) := \sum_\mu c_\mu \int_0^1 \frac{dt}{t} P_t^\mu ((Q_t^{1,\mu} f)(Q_t^{2,\mu} g))$: resonating term (need regularity)

$P^\nu, P^\mu \in StGC^{[0,b/2)}$, $Q^{1,\nu}, Q^{2,\nu}, Q^{1,\mu}, Q_t^{2,\mu} \in StGC^{[b/2,2b]}$

For any $I \subset [0,\infty)$, $StGC^I = \{((\sqrt{t}\nabla)^\alpha P_t^{(c)})_{t\in(0,1]} : \alpha \in \mathbb{Z}_+^2, \underline{\alpha_1 + \alpha_2} \in I \cap \mathbb{Z}, c \in \mathbb{N} \cap [1,b]\}$

standard families of Gaussian operators $\qquad \nwarrow$ the order of cancellation

The Besov Spaces

For $p, q \in [1, \infty], \alpha \in (-2b, 2b)$, $\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^2) = \overline{C_0^{\infty}(\mathbb{R}^2)}^{\|\cdot\|_{\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^2)}}$: the Besov Space

$$\|f\|_{\mathcal{B}^{\alpha}_{p,q}(\mathbb{R}^2)} := \|e^{\Delta} f\|_{L^p(\mathbb{R}^2:dx)} + \sum_{Q \in StGC^{(|\alpha|,2b]}} \|t^{-\alpha/2}\|Q_t f\|_{L^p(\mathbb{R}^2:dx)}\|_{L^q([0,1]:t^{-1}dt)}$$

$\mathcal{B}^{\alpha}_{\infty,\infty}(\mathbb{R}^2) =: \mathcal{C}^{\alpha}(\mathbb{R}^2)$: the Besov α-Hölder space

$\mathcal{B}^{\alpha}_{2,2}(\mathbb{R}^2) =: \mathcal{H}^{\alpha}(\mathbb{R}^2)$: the Sobolev space with the index α.

The heat semigroup approach is useful because the effect of each function in products decays exponentially. To clarify this, we introduce the partition of unity as follows:

$\{\chi_a\}_{a \in \mathbb{Z}^2} \subset C^{\infty}(\mathbb{R}^2 \to [0,1])$ s.t. $\sum_{a \in \mathbb{Z}^2} \chi_a^2 \equiv 1$, $\operatorname{supp} \chi_a \subset \square_2(a) := a + (-1,1)^2$

$\chi_a(\cdot) = \chi_0(\cdot - a)$

The continuity and the exponential decay of paraproducts

(i) For any $\alpha \in \mathbb{R}$ and $\epsilon \in (0,1)$,
$\|\chi_{a_1} P_{\chi_{a_2} g}(\chi_{a_3} f)\|_{\mathcal{H}^{\alpha-\epsilon}(\mathbb{R}^2)}$
$$\leq \begin{cases} C_{\alpha,\epsilon}\|\chi_{a_3} f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_2} g\|_{L^\infty(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \\ C_{\alpha,\epsilon}\|\chi_{a_3} f\|_{\mathcal{C}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_2} g\|_{L^2(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \end{cases}$$
(ii) For any $\alpha \in (-\infty, 0)$ and $\beta \in \mathbb{R}$,
$\|\chi_{a_1} P_{\chi_{a_2} f}(\chi_{a_3} g)\|_{\mathcal{H}^{\alpha+\beta}(\mathbb{R}^2)}$
$$\leq \begin{cases} C_{\alpha,\beta}\|\chi_{a_2} f\|_{\mathcal{C}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_3} g\|_{\mathcal{H}^{\beta}(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \\ C_{\alpha,\beta}\|\chi_{a_2} f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_3} g\|_{\mathcal{C}^{\beta}(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2)) \end{cases}$$
(iii) For any $\alpha, \beta \in \mathbb{R}$ such that $\alpha+\beta>0$,
$\|\chi_{a_1} \Pi(\chi_{a_2} f, \chi_{a_3} g)\|_{\mathcal{H}^{\alpha+\beta}(\mathbb{R}^2)}$
$\leq C_{\alpha,\beta}\|\chi_{a_2} f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)}\|\chi_{a_3} g\|_{\mathcal{C}^{\beta}(\mathbb{R}^2)} \exp(-C(|a_1-a_2|^2+|a_1-a_3|^2))$.

A modification of the operator

$$\|\chi_a \xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \le C_{\epsilon,\xi} (\log(2+|a|))^{1/2}$$

$$\Delta^{-loc} := -\int_0^1 dt\ e^{t\Delta} \text{ satisfying } \Delta^{-loc}\Delta = \Delta\Delta^{-loc} = I - e^{\Delta}$$

$$\|\chi_{a_1}\Delta^{-loc}\chi_{a_2} f\|_{\mathcal{C}^{\alpha}(\mathbb{R}^2)} \le C_{\alpha,\epsilon}\|\chi_{a_2} f\|_{\mathcal{C}^{\alpha+\epsilon-2}(\mathbb{R}^2)} \exp(-C|a_1-a_2|^2)$$

$$\|\chi_{a_1}\Delta^{-loc}\chi_{a_2} f\|_{\mathcal{H}^{\alpha}(\mathbb{R}^2)} \le C_{\alpha,\epsilon}\|\chi_{a_2} f\|_{\mathcal{H}^{\alpha+\epsilon-2}(\mathbb{R}^2)} \exp(-C|a_1-a_2|^2)$$

$\Pi(\Delta^{-loc}\xi, \xi)$in a formal expression of $H^{\xi}u = -\Delta u + \xi u$ cannot be defined, and

where an element u of the domain should be specified

we can obtain a well-defined operator $\widetilde{H^{\xi}} u$ by replacing this by a $\bigcap_{\epsilon>0} C^{-\epsilon}_{loc}(\mathbb{R}^2)$-valued random variable Y_ξ s.t. $\lim_{\varepsilon\to 0} \mathbb{E}[\|\chi_a(Y_{\xi_\varepsilon} - Y_\xi)\|^p_{C^{-\epsilon}(\mathbb{R}^2)}] = 0$ for any $a \in \mathbb{Z}^2$, $p \in [1,\infty)$ and $\epsilon > 0$, where

$$Y_{\xi_\varepsilon} := \Pi(\Delta^{-loc}\xi_\varepsilon, \xi_\varepsilon) - \underbrace{\mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon, \xi_\varepsilon)]}_{\text{diverge as } \varepsilon \to 0}$$

$$\|\chi_a Y_\xi\|_{C^{-\epsilon}(\mathbb{R}^2)} \le C_{\epsilon,\xi}\log(2+|a|)$$

Commutators and Products of 3 functions

$$C(f,g,h) := \Pi(\Delta^{-loc}P_f g, h) - f\Pi(\Delta^{-loc}g, h)$$
$$S(f,g,h) := P_h(\Delta^{-loc}P_f g) - {}_fP_h(\Delta^{-loc}g)$$
$${}_fP_h g := \sum_\nu c_\nu \int_0^1 \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu h)(Q_t^{2,\nu} g)f)$$

(i) For any $\epsilon, \alpha \in (0,1), \beta \in \mathbb{R}, \gamma \in (-\infty, 0)$ such that $\beta + \gamma < 0$ and $\alpha + \beta + \gamma > 0$,

$$\|\chi_{a_1} C(\chi_{a_2} f, \chi_{a_3} g, \chi_{a_4} h)\|_{\mathcal{H}^{\alpha+\beta+\gamma-\epsilon}(\mathbb{R}^2)}, \|\chi_{a_1} S(\chi_{a_2} f, \chi_{a_3} g, \chi_{a_4} h)\|_{\mathcal{H}^{\alpha+\beta+\gamma-\epsilon}(\mathbb{R}^2)}$$
$$\leq C_{\epsilon,\alpha,\beta,\gamma} \|\chi_{a_2} f\|_{\mathcal{H}^\alpha(\mathbb{R}^2)} \|\chi_{a_3} g\|_{\mathcal{C}^{\beta-2}(\mathbb{R}^2)} \|\chi_{a_4} h\|_{\mathcal{C}^\gamma(\mathbb{R}^2)}$$
$$\times \exp(-C(|a_1 - a_2|^2 + |a_1 - a_3|^2 + |a_1 - a_4|^2))$$

(ii) For any $\alpha \in (-\infty, 0)$, $\beta \in \mathbb{R}$ and $\epsilon \in (0,1)$,

$$\|\chi_{a_1} {}_{\chi_{a_2} h}P_{\chi_{a_3} f}(\chi_{a_4} g)\|_{\mathcal{H}^{\alpha+\beta-\epsilon}(\mathbb{R}^2)}$$
$$\leq C_{\alpha,\beta,\epsilon} \|\chi_{a_3} f\|_{\mathcal{C}^\alpha(\mathbb{R}^2)} \|\chi_{a_4} g\|_{\mathcal{C}^\beta(\mathbb{R}^2)} \|\chi_{a_2} h\|_{L^2(\mathbb{R}^2)}$$
$$\times \exp(-C(|a_1 - a_2|^2 + |a_2 - a_3|^2 + |a_2 - a_4|^2))$$

Our operator $\widetilde{H^\xi}$

$$\Phi_\xi(u) := u - \Delta^{-loc} P_u \xi - \Delta^{-loc}{}_u P_\xi(\Delta^{-loc}\xi) - \Delta^{-loc} P_u Y_\xi$$

$$\mathrm{Dom}_{+0}(\widetilde{H^\xi}) := \Big\{ u \in \bigcap_{\epsilon>0} \mathcal{H}^{1-\epsilon}(\mathbb{R}^2) : \limsup_{|a|\to\infty} \frac{1}{|a|} \log \|\chi_a u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} < 0 \text{ for any } \epsilon > 0,$$

$$\Phi_\xi(u) \in \mathcal{H}^2(\mathbb{R}^2), \limsup_{|a|\to\infty} \frac{1}{|a|} \log \|\chi_a \Phi_\xi(u)\|_{\mathcal{H}^2(\mathbb{R}^2)} < 0 \Big\},$$

$$\begin{aligned}\widetilde{H^\xi} u := & -\Delta\Phi_\xi(u) + P_\xi \Phi_\xi(u) + \Pi(\Phi_\xi(u), \xi) + P_1^{(b)}((P_1^{(b)} u)(P_1^{(b)} \xi)) \\ & + e^\Delta P_u \xi + e^\Delta{}_u P_\xi(\Delta^{-loc}\xi) + e^\Delta P_u Y_\xi + C(u, \xi, \xi) + S(u, \xi, \xi) \\ & + P_{Y_\xi} u + \Pi(u, Y_\xi) + P_1^{(b)}((P_1^{(b)} u)(P_1^{(b)} Y_\xi)) \\ & + P_\xi(\Delta^{-loc}{}_u P_\xi(\Delta^{-loc}\xi)) + \Pi(\Delta^{-loc}{}_u P_\xi(\Delta^{-loc}\xi), \xi) \\ & + P_\xi(\Delta^{-loc} P_u Y_\xi) + \Pi(\Delta^{-loc} P_u Y_\xi, \xi).\end{aligned}$$

An abstract representation of the operator $\widetilde{H^\xi}$

$$
\begin{aligned}
\widetilde{H^\xi} u \sim -\Delta \Phi_\xi(u) + \sum \int_0^1 \frac{dt}{t} \int dx_1 \, O\Big(\frac{1}{t} \exp\Big(-\frac{|x-x_1|^2}{t}\Big)\Big) u(x_1) \\
\times \int dx_2 \, O\Big(\frac{1}{t} \exp\Big(-\frac{|x-x_2|^2}{t}\Big)\Big) \begin{matrix} \xi(x_2) \\ Y_\xi(x_2) \end{matrix} \\
\times \Big(\int dx_3 \, O\Big(\frac{1}{t} \exp\Big(-\frac{|x-x_3|^2}{t}\Big)\Big) \xi(x_3) \Big)
\end{aligned}
$$

Main Statements

Theorem (Self-adjointness, U(Stochastic Processes and their Application, 2025))

The operator $(\widetilde{H^\xi}, \mathrm{Dom}_{+0}(\widetilde{H^\xi}))$ *is essentially self-adjoint on* $L^2(\mathbb{R}^2)$.

We denote the unique self-adjoint extension by the same symbol $\widetilde{H^\xi}$.

Theorem (Spectrum, U(Stochastic Processes and their Application, 2025))

The spectral set of $\widetilde{H^\xi}$ *is* $\mathbb{R}$.

Theorem (Exponential Localization, U(New))

$\exists E_0 \in (-\infty, 0)$ *s. t. for a. a.* ξ, $(-\infty, E_0] \subset \mathrm{spec}_{pp}(\widetilde{H^\xi})$ *and any corresponding eigenfunction* ϕ_ξ *satisfies*

$$\varlimsup_{|a|\to\infty} |a|^{-1} \log \|\chi_a \phi_\xi\|_{L^2(\mathbb{R}^2)} < 0.$$

A useful tool to treat the operator $\widetilde{H^\xi}$

Smooth approximation $\widetilde{H^{\xi_\varepsilon}} = -\Delta + \xi_\varepsilon - \mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon, \xi_\varepsilon)]$
is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ since $\underset{\text{smooth}}{|\xi_\varepsilon(x)|} \leq C_{\xi,\varepsilon}(\log(2+|x|))^{1/2}$

But $C_0^\infty(\mathbb{R}^2) \not\subset \mathrm{Dom}_{+0}(\widetilde{H^\xi})$ since $\Phi_\xi(C_0^\infty(\mathbb{R}^2)) \not\subset \mathcal{H}^2(\mathbb{R}^2)$

$\mathrm{Dom}_{+0}(\widetilde{H^\xi})$ depends on ξ

Our useful tool is

$$\Phi_\xi^s(u) := u - \sum_{a\in\mathbb{Z}^2} \Delta^{-loc} P_u^{s(a)}(\chi_a^2\xi) - \sum_{a,a'\in\mathbb{Z}^2} \Delta^{-loc}\, {}_u P_{\chi_a^2\xi}^{s_1(a,a')}(\Delta^{-loc}\chi_{a'}^2\xi) - \sum_{a\in\mathbb{Z}^2} \Delta^{-loc} P_u^{s_2(a)}(\chi_a^2 Y_\xi)$$

$$P_f^s g := \sum_\nu c_\nu \int_0^s \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g))$$

$$ {}_h P_f^s g := \sum_\nu c_\nu \int_0^s \frac{dt}{t} Q_t^{1,\nu}((P_t^\nu f)(Q_t^{2,\nu} g) h)$$

Choice of s

For any $\epsilon \in (0,1)$ and almost all ξ,
$\mathbf{s}(\epsilon,\xi,\delta) = (s(a;\epsilon,\xi,\delta), s_1(a,a';\epsilon,\xi,\delta), s_2(a;\epsilon,\xi,\delta))_{a\in\mathbb{Z}^2}$ is taken so that

$$\|\chi_a(I - \Phi_\xi^{\mathbf{s}(\epsilon,\xi,\delta)})(u)\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \le \delta \sum_{a'\in\mathbb{Z}^2} \exp(-M|a-a'|^2)\|\chi_{a'}u\|_{L^2(\mathbb{R}^2)}$$

Indeed, there exist $s(\epsilon,\xi), s_1(\epsilon,\xi), s_2(\epsilon,\xi) \in (0,1)$ and
$M, M(\epsilon), M_1(\epsilon), M_2(\epsilon) \in (0,\infty)$ s.t.

$$s(a;\epsilon,\xi,\delta) = s(\epsilon,\xi)\left(\frac{\delta}{(\log(2+|a|))^2}\right)^{M(\epsilon)},$$

$$s_1(a,a';\epsilon,\xi,\delta) = s_1(\epsilon,\xi)\left(\frac{\delta}{(\log(2+|a|))^2(\log(2+|a'|))^2}\right)^{M_1(\epsilon)}$$

$$s_2(a;\epsilon,\xi,\delta) = s_2(\epsilon,\xi)\left(\frac{\delta}{\log(2+|a|)}\right)^{M_2(\epsilon)}.$$

Inverse of Φ_ξ^s

$$\|(I-\Phi_\xi^{s(\epsilon,\xi,\delta)})(u)\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \le C_{\xi,\epsilon}\delta\|u\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}$$

Thus for $\delta \in (0, 1/C_{\xi,\epsilon})$, there exists the inverse $(\Phi_\xi^{s(\epsilon,\xi,\delta)})^{-1} = \sum_{n=0}^{\infty}(I-\Phi_\xi^{s(\epsilon,\xi,\delta)})^n$

s.t. $\|(\Phi_\xi^{s(\epsilon,\xi,\delta)})^{-1}(v)\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)} \le \|v\|_{\mathcal{H}^{1-\epsilon}(\mathbb{R}^2)}/(1-C_{\xi,\epsilon}\delta)$

$(\Phi_\xi^{s(\epsilon,\xi,\delta)})^{-1}(\{v \in \mathcal{H}^2(\mathbb{R}^2) : \text{supp}\, v \text{ is compact}\}) \subset \text{Dom}_{+0}(\widetilde{H^\xi})$

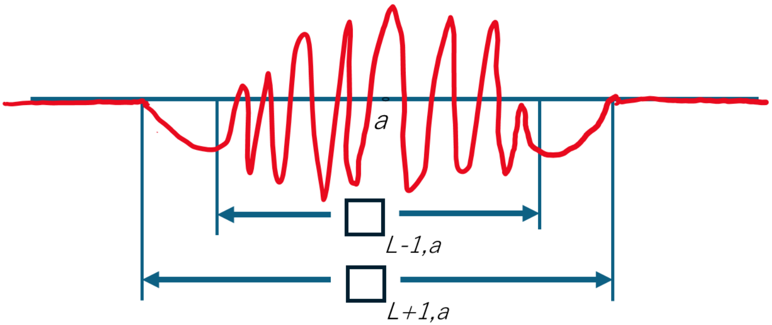
since $\Phi_\xi - \Phi_\xi^{s(\epsilon,\xi,\delta)}$ is smooth and has an exponenially decaying property.
By this we can take many elements of the domain.
This is the most important point I get from the work by Mousard.

Restrict the white noise to the square $\square_{L,\boldsymbol{a}} := \boldsymbol{a} + (-L/2, L/2)^2$

$\widetilde{\xi} = (\xi, \overline{\xi})$ where $\overline{\xi} = (\overline{\xi_a})_{a \in \mathbb{Z}^2}$ i.i.d. bdd, with C_0^∞ density indep. of ξ

$$\xi \to \widetilde{\xi}_{L,\boldsymbol{a}} := \underset{\sum_{a \in \mathbb{Z}^2 \cap \square_{L-2,\boldsymbol{a}}} \chi_a^2 \xi}{\underset{\|}{\xi_{L-2,\boldsymbol{a}}}} + \underset{\sum_{a \in \mathbb{Z}^2 \cap (\square_{L,\boldsymbol{a}} \setminus \square_{L-2,\boldsymbol{a}})} \chi_a^2 \overline{\xi}_a}{\underset{\|}{\overline{\xi_{L,\boldsymbol{a}}}}}$$

Omit $\boldsymbol{a}$ if $\boldsymbol{a} = 0$



In $\widetilde{H^\xi}$, $\xi \to \widetilde{\xi}_{L,a}$

For $u \in \mathrm{Dom}(\widetilde{H^{\widetilde{\xi}}_{L,\boldsymbol{a}}}) = \left\{ u \in \bigcap_{\epsilon>0} \mathcal{H}^{1-\epsilon}(\mathbb{R}^2) : \Phi_{\xi,L-2,\boldsymbol{a}}(u) \in \mathcal{H}^2(\mathbb{R}^2) \right\}$,

$$\widetilde{H^{\widetilde{\xi}}_{L,\boldsymbol{a}}} u = \underset{\parallel}{\widetilde{H^{\xi}_{L-2,\boldsymbol{a}}}} u + \overline{\xi_{L,\boldsymbol{a}}} u$$

$$-\Delta\Phi_{\xi,L-2,\boldsymbol{a}}(u) + P_{\xi_{L-2,\boldsymbol{a}}}\Phi_{\xi,L-2,\boldsymbol{a}}(u) + \Pi(\Phi_{\xi,L-2,\boldsymbol{a}}(u), \xi_{L-2,\boldsymbol{a}})$$
$$+P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}\xi_{L-2,\boldsymbol{a}}))$$
$$+e^{\Delta}P_u\xi_{L-2,\boldsymbol{a}} + e^{\Delta}{}_uP_{\xi_{L-2,\boldsymbol{a}}}(\Delta^{-loc}\xi_{L-2,\boldsymbol{a}}) + e^{\Delta}P_uY_{\xi,L-2,\boldsymbol{a}} + C(u, \xi_{L-2,\boldsymbol{a}}, \xi_{L-2,\boldsymbol{a}})$$
$$+S(u, \xi_{L-2,\boldsymbol{a}}, \xi_{L-2,\boldsymbol{a}}) + P_{Y_{\xi,L-2,\boldsymbol{a}}}u + \Pi(u, Y_{\xi,L-2,\boldsymbol{a}}) + P_1^{(b)}((P_1^{(b)}u)(P_1^{(b)}Y_{\xi,L-2,\boldsymbol{a}}))$$
$$+P_{\xi_{L-2,\boldsymbol{a}}}(\Delta^{-loc}{}_uP_{\xi_{L-2,\boldsymbol{a}}}(\Delta^{-loc}\xi_{L-2,\boldsymbol{a}})) + \Pi(\Delta^{-loc}{}_uP_{\xi_{L-2,\boldsymbol{a}}}(\Delta^{-loc}\xi_{L-2,\boldsymbol{a}}), \xi_{L-2,\boldsymbol{a}})$$
$$+P_{\xi_{L-2,\boldsymbol{a}}}(\Delta^{-loc}P_uY_{\xi,L-2,\boldsymbol{a}}) + \Pi(\Delta^{-loc}P_uY_{\xi,L-2,\boldsymbol{a}}, \xi_{L-2,\boldsymbol{a}})$$

$$Y_{\xi,L-2,\boldsymbol{a}} = \lim_{\varepsilon\to 0}(\Pi(\Delta^{-loc}\sum_{a\in\mathbb{Z}^2\cap\square_{L-2,\boldsymbol{a}}} \chi_a^2 e^{\varepsilon^2\Delta}\xi, \sum_{a\in\mathbb{Z}^2\cap\square_{L-2,\boldsymbol{a}}} \chi_a^2 e^{\varepsilon^2\Delta}\xi) - \mathbb{E}[\Pi('')])$$

$$\Phi_{\xi,L-2,\boldsymbol{a}}(u) = u - \Delta^{-loc}\{P_u\xi_{L-2,\boldsymbol{a}} +_u P_{\xi_{L-2,\boldsymbol{a}}}(\Delta^{-loc}\xi_{L-2,\boldsymbol{a}}) + P_uY_{\xi,L-2,\boldsymbol{a}}\}$$

Properties of the operator with the restricted whitenoise

$k_{\xi,L-2}$: a positive polynomial of $(\|\xi_{L-2}\|_{C^{-1-\epsilon}(\mathbb{R}^2)}, \|Y_{\xi,L-2}\|_{C^{-\epsilon}(\mathbb{R}^2)})$ depend on L:

$\|\nabla\Phi_{\xi,L-2}(u)\|^2_{L^2(\mathbb{R}^2)} \leq (u, (\widetilde{H_L^{\tilde{\xi}}} + k_{\xi,L-2}))u)_{L^2(\mathbb{R}^2)}$

We can show that $\mathrm{Ran}(\widetilde{H_L^{\tilde{\xi}}} + k_{\xi,L-2}) = L^2(\mathbb{R}^2)$

Lemma (Self-adjointness of the operator with the restricted whitenoise)

The operator $\widetilde{H_L^{\tilde{\xi}}}$ with the domain $\mathrm{Dom}(\widetilde{H_L^{\tilde{\xi}}})$ is self-adjoint on $L^2(\mathbb{R}^2)$.

Proof of Theorem on the self-adjointness

For $\forall f \in \mathrm{Ran}(\widetilde{H^\xi} + i)^\perp$,

$\|f\|^2_{L^2(\mathbb{R}^2)} = \lim_{R\to\infty} (f, \widetilde{\chi_R} f)_{L^2(\mathbb{R}^2)}$, where $\widetilde{\chi_R}$ is C^∞, $= 1$ on $\square_{R-1}$, $= 0$ on $\square_R^c$.

$\widetilde{\chi_R} f = (\widetilde{H^{\tilde{\xi}}_{R+L}} + i)\varphi_{R,L}$ with $\exists \varphi_{R,L} \in \mathrm{Dom}(\widetilde{H^{\tilde{\xi}}_{R+L}})$ and $L > 0$ by the S.A. of $\widetilde{H^{\tilde{\xi}}_{R+L}}$

$\varphi_{R,L}$ is near to $\widetilde{\varphi_{R,L}} := (\Phi_\xi^{s(\epsilon,\xi,\delta)})^{-1}(\Phi^{s(\epsilon,\xi,\delta)}_{\xi,R+L-2}(\varphi_{R,L})) \in \mathrm{Dom}_{+0}(\widetilde{H^\xi})$

Since $(\widetilde{H^\xi} + i)\widetilde{\varphi_{R,L}} \in \mathrm{Ran}(\widetilde{H^\xi} + i)$ is orthogonal to f, we have

$$\|f\|^2_{L^2(\mathbb{R}^2)} = \lim_{R\to\infty} (f, \underbrace{(\widetilde{H^{\tilde{\xi}}_{R+L}} + i)\varphi_{R,L} - (\widetilde{H^\xi} + i)\widetilde{\varphi_{R,L}}}_{\downarrow \text{ as } L\to\infty \text{ owing to good estimates of } \Phi^s_\xi \atop 0})_{L^2(\mathbb{R}^2)} = 0$$

$\therefore \overline{\mathrm{Ran}(\widetilde{H^\xi} + i)} = L^2(\mathbb{R}^2)$

Resolvent convergence

For $\widetilde{H^{\xi}}$ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$,

$$\sup_{\|v\|_{L^2(\mathbb{T}^2)}=1} \|(\widetilde{H^{\xi_\varepsilon}} + z)^{-1}v - (\widetilde{H^{\xi}} + z)^{-1}v\|_{L^2(\mathbb{T}^2)} \xrightarrow{\varepsilon\to 0} 0 \text{ for sufficiently large } z \in \mathbb{R}$$

(Allez-Chouk Th.1.6, Mouzard Prop.2.14)

$\lambda_n(\widetilde{H^{\xi_\varepsilon}}) \xrightarrow{\varepsilon\to 0} \lambda_n(\widetilde{H^{\xi}})$ (Allez-Chouk Th.1.6, Mouzard Cor.2.15)

For $\widetilde{H^{\xi}}$ on $\mathbb{R}^2$, (U(2025) Prop.4.1)

$\|(\widetilde{H^{\xi_\varepsilon}} + z)^{-1}v - (\widetilde{H^{\xi}} + z)^{-1}v\|_{L^2(\mathbb{R}^2)} \xrightarrow{\varepsilon\to 0} 0$ for each $v \in L^2(\mathbb{R}^2)$ and $z \in \mathbb{C} \setminus \mathbb{R}$

However the estimate with $\sup\limits_{\|v\|_{L^2(\mathbb{R}^2)}=1}$ may be difficult.

For the identification of the spectrum, we use the method used for stationary random operators.

For $\forall r \in \mathbb{R}$ and $L >> 0$, $\exists$ event s.t. $\xi \risingdotseq r$ on $\square_{L/2}$

$$E(\varepsilon, r, L) = \Big\{ \xi : \ |X_{\mathbf{0}}(\xi^L) - r|, |X_{\mathbf{n}}(\xi^L)| \le \varepsilon / L^{20} \text{ for } \mathbf{n} \in \mathbb{Z}^2 \cap \square_{L^{10}} \setminus \{\mathbf{0}\},$$

$$\left. \begin{array}{c} \|\chi_a \widetilde{\xi^L}\|_{\mathcal{C}^{-1-\epsilon}(\mathbb{R}^2)}, \\ \|\chi_a Y(\widetilde{\xi^L})\|_{\mathcal{C}^{-\epsilon}(\mathbb{R}^2)} \end{array} \right\} \le \left\{ \begin{array}{c} 1/L \ (a \in \mathbb{Z}^2 \cap \square_L) \\ |a| \ (a \in \mathbb{Z}^2 \setminus \square_L) \end{array} \right\},$$

in terms of a Fourier series representation

$$\xi = \widetilde{\chi_L} \sum_{\mathbf{n} \in \mathbb{Z}^2 \cap \square_{L^{10}}} X_{\mathbf{n}}(\xi^L) \varphi^L_{\mathbf{n}}(x) + \widetilde{\xi^L}$$

Partial Fourier sum

$\{X_{\mathbf{n}}(\xi^L)\} \underset{\text{i.i.d.}}{\sim} \mathrm{N}(0,1)$

$\{\varphi^L_{\mathbf{n}}\}$ is ONS of $L^2(\square_L)$

$\varphi^L_0 \equiv 1/L$

independent remaining terms

$Y(\widetilde{\xi^L})$: $\bigcap_{\epsilon>0} C^{-\epsilon}_{loc}(\mathbb{R}^2)$-valued r.v.

obtained by $\widetilde{\xi^L} \to \xi$ in Y_ξ

$\exists$ a function that constitutes a Weyl sequence under the event

$$\forall \lambda \in \mathbb{R}, \varepsilon > 0, L >> 0, \exists r, c(\lambda, L) \in \mathbb{R} \; \forall \xi \in E(\varepsilon, r, L) \; \exists \varphi \in C_0^\infty(\square_{L/2})$$

s.t. $\|(\widetilde{H^\xi} - \lambda) \underset{\underset{(\Phi^{s(\epsilon,\xi,L,\delta)}_{\xi,L,N<})^{-1}(\varphi_{\varepsilon,R})}{\|}}{\widetilde{\varphi}} \|_{L^2(\mathbb{R}^2)} < c\,(\lambda, L)\varepsilon, \; \|\varphi\| = 1$

$\mathbb{P}(E(\varepsilon, r, L)) > 0$ for any r, L.

Proof of Theorem on the spectral identification by the ergodicity

Let $E(x_0, \varepsilon, r, L) := \{\xi : \xi(\cdot - x_0) \in E(\varepsilon, r, L)\}$
Then $\bigcup_{x_0 \in \mathbb{Z}^2} E(x_0, \varepsilon, r, L)$ is $\mathbb{Z}^2$-invariant.
By the ergodicity of the white noise, we have

$$\mathbb{P}\Big(\bigcup_{x_0 \in \mathbb{Z}^2} E(x_0, \varepsilon, r, L) \Big) = 1.$$

By the shift, we can take a Weyl sequence with probability 1.
Thus $\lambda \in \mathrm{Spec}(\widetilde{H^\xi})$,
and we obtain $\mathrm{Spec}(\widetilde{H^\xi}) = \mathbb{R}$.

The negative spectrum of the operator with the restricted noises

$\widetilde{H^{\tilde{\xi}}_{L,\mathbf{a}}}$ is the norm resolvent limit of $\widetilde{H^{\tilde{\xi}_\varepsilon}_{L,\mathbf{a}}} := \widetilde{H^{\xi_\varepsilon}_{L-2,\mathbf{a}}} + \bar{\xi}_{L,\mathbf{a}}$ as $\varepsilon \to 0$

The negative spectra of the operator $\widetilde{H^{\tilde{\xi}}_{L,\mathbf{a}}}$ are discrete.

Lemma (Moments of numbers of negative eigenvalues)

$\forall \lambda > 0$, $\forall p \geq 1$, $\exists c_{\lambda,p,1}, \exists c_{\lambda,p,2} \in (0,\infty)$ *s.t.*

$\mathbb{E}[Tr[1_{(-\infty,-\lambda]}(\widetilde{H^{\tilde{\xi}}_{L,\mathbf{a}}})]^p]^{1/p} \leq c_{\lambda,p,1} L^{c_{\lambda,p,2}}$ *for* $\forall L \in 2\mathbb{N}$.

$$\because\ \mathbb{E}[\mathrm{Tr}[1_{(-\infty,-\lambda]}(\widetilde{H^{\tilde{\xi}}_{L}})]^p]^{1/p} \leq \varliminf_{\varepsilon\to 0} \mathbb{E}[\mathrm{Tr}[1_{(-\infty,-\lambda]}(\widetilde{H^{\tilde{\xi}_\varepsilon}_{L}})]^p]^{1/p}$$

Since $\widetilde{H^{\tilde{\xi}_\varepsilon}_{L}}$ is a relatively compact perturbation of $-\Delta$, we apply the Birman-Schwinger principle.

To address the singularity of ξ, we replace $-\Delta$ by $\widehat{H^{\tilde{\xi}_\varepsilon,s}_{L-2}}$ in the next slide:

An operator with the restricted noise which is near to ≥ 0

For $u \in \mathrm{Dom}(\widehat{H^{\xi,s}_{L-2}}) = \left\{ u \in \bigcap_{\epsilon>0} \mathcal{H}^{1-\epsilon}(\mathbb{R}^2) : \Phi^s_{\xi,L-2}(u) \in \mathcal{H}^2(\mathbb{R}^2) \right\}$,

$$
\begin{aligned}
\widehat{H^{\xi,s}_{L-2}} u = & -\Delta \Phi^s_{\xi,L-2}(u) + P^s_{\xi_{L-2}}(\Phi^s_{\xi,L-2}(u)) + \Pi^s(\Phi^s_{\xi,L-2}(u), \xi_{L-2}) \\
& + e^{s\Delta} P^s_u \xi_{L-2} + e^{s\Delta}{}_u P^s_{\xi_{L-2}}(\Delta^{-loc} \xi_{L-2}) + e^{s\Delta} P^s_u Y^s_{\xi,L-2} \\
& + C^s(u, \xi_{L-2}, \xi_{L-2}) + S^s(u, \xi_{L-2}, \xi_{L-2}) \\
& + P^s_{Y^s_{\xi,L-2}} u + \Pi^s(u, Y^s_{\xi,L-2}) + P^{(b)}_s((P^{(b)}_s u)(P^{(b)}_s Y^s_{\xi,L-2})) \\
& + P^s_{\xi_{L-2}}(\Delta^{-loc}{}_u P^s_{\xi_{L-2}}(\Delta^{-loc} \xi_{L-2})) + \Pi^s(\Delta^{-loc}{}_u P^s_{\xi_{L-2}}(\Delta^{-loc} \xi_{L-2}), \xi_{L-2}) \\
& + P^s_{\xi_{L-2}}(\Delta^{-loc} P^s_u Y^s_{\xi,L-2}) + \Pi^s(\Delta^{-loc} P^s_u Y^s_{\xi,L-2}, \xi_{L-2})
\end{aligned}
$$

An application of the Birman-Schwinger principle

For $\forall \lambda > 0$,
$s(\xi, \lambda, L-2) := c_1 \lambda^{c_2} / (1 + \|\xi_{L-2}\|^2_{C^{-1-\epsilon}(\mathbb{R}^2)} + \sup_{s \in (0,1]} \| Y^s_{\xi, L-2} \|_{C^{-\epsilon}(\mathbb{R}^2)})^{c_3}$
$\Rightarrow \widehat{H_{L-2}^{\xi, s(\xi,\lambda,L-2)}} \geq -\lambda/4$

$\widetilde{H^{\xi}_{L-2}} u = \widehat{H^{\xi,s}_{L-2}} u + P^{(b)}_s ((P^{(b)}_s u)(P^{(b)}_s \xi_{L-2})) - \overline{Y^s_{L-2}} u$
with $\overline{Y^s_{L-2}} := \mathbb{E}[(\Pi - \Pi^s)(\Delta^{-loc} \xi_{L-2}, \xi_{L-2})]$

By the Birman-Schwinger principle,

$$\mathrm{Tr}[1_{(-\infty,-\lambda]}(\widetilde{H^{\xi_\varepsilon}_{L-2}} + \overline{\xi}_L)] \leq \mathrm{Tr}[1_{[1,\infty)}(\Gamma^{(\widetilde{\xi},\varepsilon)})] \leq \mathrm{Tr}[(\Gamma^{(\widetilde{\xi},\varepsilon)})^2] = \|\Gamma^{(\widetilde{\xi},\varepsilon)}\|^2_{\mathcal{I}_2},$$

Our Birman-Schwinger kernel

$$\Gamma^{(\widetilde{\xi},\varepsilon)} = -\Gamma_0^{(\xi,\varepsilon)} + \Gamma_1^{(\xi,\varepsilon)} - \Gamma_2^{(\widetilde{\xi},\varepsilon)},$$

where

$$\begin{aligned}
\Gamma_0^{(\xi,\varepsilon)} =& (H_{L-2}^{\xi_\varepsilon,\widehat{s(\xi,\lambda,L}-2)} + \lambda)^{-1/2}(P_{s(\xi,\lambda,L-2)}^{(b)}((P_{s(\xi,\lambda,L-2)}^{(b)}\xi_{\varepsilon,L-2})(P_{s(\xi,\lambda,L-2)}^{(b)}\cdot))) \\
& \times (H_{L-2}^{\xi_\varepsilon,\widehat{s(\xi,\lambda,L}-2)} + \lambda)^{-1/2}, \\
\Gamma_1^{(\xi,\varepsilon)} =& (H_{L-2}^{\xi_\varepsilon,\widehat{s(\xi,\lambda,L}-2)} + \lambda)^{-1/2}\overline{Y_{L-2}^{s(\xi,\lambda,L-2)}}(H_{L-2}^{\xi_\varepsilon,\widehat{s(\xi,\lambda,L}-2)} + \lambda)^{-1/2}, \\
\Gamma_2^{(\widetilde{\xi},\varepsilon)} =& (H_{L-2}^{\xi_\varepsilon,\widehat{s(\xi,\lambda,L}-2)} + \lambda)^{-1/2}\overline{\xi}_L(H_{L-2}^{\xi_\varepsilon,\widehat{s(\xi,\lambda,L}-2)} + \lambda)^{-1/2}.
\end{aligned}$$

$\|\Gamma_1^{(\xi,\varepsilon)}\|_{\mathcal{I}_2} \lesssim \|\Gamma_{1,1}^{(\xi,\varepsilon)}\|_{\mathcal{I}_2} + \|\Gamma_{1,2}^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}$

$k_{\xi,L-2}$: a positive polynomial of $\|\xi_{L-2}\|_{C^{-1-\epsilon}(\mathbb{R}^2)}$ and $\|Y_{\xi,L-2}\|_{C^{-\epsilon}(\mathbb{R}^2)}$ s.t.

$$\|u\|^2_{L^2(\mathbb{R}^2)} \leq (u, (\widetilde{H^{\xi_\varepsilon}_{L-2}} + k_{\xi,L-2})u)_{L^2(\mathbb{R}^2)} \text{ for } \forall\varepsilon \in 0,1)$$

By $(\widetilde{H^{\xi_\varepsilon}_{L-2}} + k_{\xi,L-2})^{-1}$

$$= \int_0^T \frac{dt}{2} \exp\left(-\frac{t}{2}\widetilde{H^{\xi_\varepsilon}_{L-2}} - \frac{t}{2}k_{\xi,L-2}\right) + \exp\left(-\frac{T}{2}\widetilde{H^{\xi_\varepsilon}_{L-2}} - \frac{T}{2}k_{\xi,L-2}\right)(\widetilde{H^{\xi_\varepsilon}_{L-2}} + k_{\xi,L-2})^{-1},$$

for any $T \in (0,\infty)$, it is enough to estimate $\|\Gamma_{1,1}^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}$ and $\Gamma_{1,2}^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}$, where

$$\Gamma_{1,1}^{(\xi,\varepsilon)} = \overline{Y_{L-2}^{s(\xi,\lambda,L-2)}} \int_0^T \frac{dt}{2} \exp\left(-\frac{t}{2}\widetilde{H^{\xi_\varepsilon}_{L-2}}\right) \exp\left(-\frac{t}{2}k_{\xi,L-2}\right)$$

$$\Gamma_{1,2}^{(\xi,\varepsilon)} = \overline{Y_{L-2}^{s(\xi,\lambda,L-2)}} \exp\left(-\frac{T}{2}\widetilde{H^{\xi_\varepsilon}_{L-2}}\right) \exp\left(-\frac{T}{2}k_{\xi,L-2}\right)$$

A moment

By applying the Feynman-Kac formula and omitting $k_{\xi,L}$,

$$\begin{aligned}
&\mathbb{E}^{\xi}[\|\Gamma^{(\xi,\varepsilon)}_{1,1}\|^2_{\mathcal{I}_2}] \\
&\leq \int_{(\mathbb{R}^2)^2} dx\ \overline{Y^{s(\xi,\lambda,L-2)}_{L-2}}(x)^2 \int_{[0,T]^2} \frac{dtd\underline{t}}{8\pi(t+\underline{t})} \\
&\quad \times \mathbb{E}^{\xi,w}\left[\exp\left(-\int_0^{t+\underline{t}} \frac{dt'}{2}(\xi_{\varepsilon,L-2}-\mathbb{E}^{\xi}[\Pi(\Delta^{-loc}\xi_{\varepsilon,L-2},\xi_{\varepsilon,L-2})])(x+w_0^{t+\underline{t}}(t'))\right)\right] \\
&\leq \int_{(\mathbb{R}^2)^2} dx\ \overline{Y^{s(\xi,\lambda,L-2)}_{L-2}}(x)^2 \int_{[0,T]^2} \frac{dtd\underline{t}}{8\pi(t+\underline{t})} \mathbb{E}^{w}\left[\exp\left(\frac{1}{2}\chi_{\varepsilon}(t+\underline{t},x,L-2,w_0^{t+\underline{t}})\right)\right],
\end{aligned}$$

where $w_0^{t+\underline{t}}$ is a 2D Brownian bridge s.t.$w_0^{t+\underline{t}}(0)=w_0^{t+\underline{t}}(t+\underline{t})=0$.

Renormalized intersection local time restricted to a finite square

$$\chi_\varepsilon(t,x,L-2,w_0^t) = 2\chi_\varepsilon^0(t,x,L-2,w_0^t) + 4\chi_\varepsilon^{bdd}(t,x,L-2,w_0^t), \text{ where}$$

$$\chi_\varepsilon^0(t,x,L-2,w_0^t) := \iint_{0\le s_1\le s_2\le t} ds_1 ds_2 \int dy e^{\varepsilon^2\Delta}(y)\underset{\sum_{a\in\mathbb{Z}^2\cap\square_{L-2}}\chi_a^2}{\underset{||}{\widetilde{\chi}_{L-2}^2}}(x+w_0^t(s_1)+y)$$

$$\times\{e^{\varepsilon^2\Delta}(y+w_0^t(s_1)-w_0^t(s_2)) - \mathbb{E}^w[e^{\varepsilon^2\Delta}(y+w_0^t(s_1)-w_0^t(s_2))]\},$$

$$\sup_{t\in[0,1],x\in\mathbb{R}^2,\varepsilon\in(0,1],L-2\in\mathbb{N},w} |\chi_\varepsilon^{bdd}(t,x,L-2,w)| < \infty$$

$$\lim_{\varepsilon\to0}\chi_\varepsilon^0(t,x,L-2,w_0^t) = \iint_{0\le s_1\le s_2\le t} ds_1 ds_2 \widetilde{\chi}_{L-2}^2(x+w_0^t(s_1))\{\delta(w_0^t(s_1)-w_0^t(s_2)) - \mathbb{E}^w[\delta(w_0^t(s_1)-w_0^t(s_2))]\}$$

formally

Results on renormalized intersection local time

$$\chi_0^0(t,x,\infty,w_0^t) = \iint_{0\le s_1\le s_2\le t} ds_1 ds_2 \{\delta(w_0^t(s_1)-w_0^t(s_2))-\mathbb{E}^w[\delta(w_0^t(s_1)-w_0^t(s_2))]\}$$

X. Chen, Mathematical Surveys and Monographs, vol. 157, AMS (2010)

$$\sup_{\varepsilon} \mathbb{E}^w[\exp(\chi_\varepsilon^0(t,x,\infty,w_0))] < \infty \text{ for small enough } t > 0,$$

where w_0 is the Brownian motion starting at 0.

T. Matsuda, Stochastic Processes and their Applications **153** (2022), 91-–127

$$\sup_{\varepsilon} \mathbb{E}^w[\exp(\chi_\varepsilon^0(t,x,\infty,w_0^t))] < \infty \text{ for small enough } t > 0$$

Proof of Lemma on the numbers of negative eigenvalues

By the same method, we have
$\sup_{\varepsilon} \mathbb{E}^{w}\left[\exp\left(\frac{1}{2}\chi_{\varepsilon}(t, x, L-2, w_0^t)\right)\right] < \infty$ for small enough $t > 0$
We also use

$$|\overline{Y_{L-2}^{s(\xi,\lambda,L-2)}}| \leq c_1 \exp(-c_2 d(x, \Box_L)^2 / s(\xi, \lambda, L-2)) \log(1/s(\xi, \lambda, L-2))$$

and take $T > 0$ sufficiently small to obtain
$\mathbb{E}^{\xi}[\|\Gamma_{1,1}^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}^2] < \infty$
Similarly we obtain
$\mathbb{E}^{\xi}[\|\Gamma_{j}^{(\xi,\varepsilon)}\|_{\mathcal{I}_2}^{2p}] < \infty$ for $j \in \{0, 1, 2\}$ and $p \in \mathbb{N}$.
Thus we can complete the proof of the lemma on the moments of numbers on negative eigenvalues.

Outline of a traditional proof of the localization

I. Initial estimate : for some $L_0 \in \mathbb{N}$

$$\|\chi_x(\widetilde{H_{L_0}^{\tilde{\xi}}} - E)^{-1}\chi_y\| \lesssim \exp(-m|x-y|)$$

by Combes-Thomas type estimate

II(Multi Scale Analysis).

Similar estimate for $L_1 > L_0$

Similar estimate for $L_2 > L_1 \cdots$

by Geometric resolvent inequality

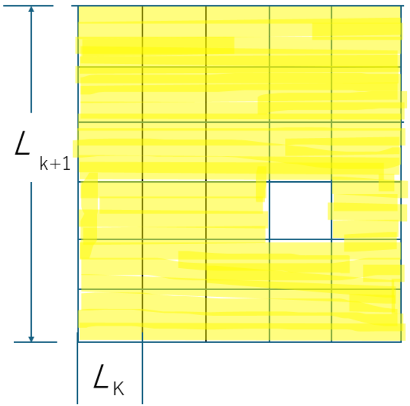
and Wegner type estimate

III. Apply the preceding estimates

to generalized eigenfunctions

(from a generalized eigenfunction expansion)

by eigenfunction decay inequality



Combes-Thomas type estimate

$$\|\chi_{a_1}(\widetilde{H^{\widetilde{\xi}}_{L_0}} - E)^{-1}\chi_{a_2}\|_{op}$$

$$\leq \frac{3}{d(E, \mathrm{spec}\widetilde{H^{\widetilde{\xi}}_{L_0}})}$$

$$\times \exp\left(\frac{-(|a_1 - a_2| - 2\sqrt{2})_+ d(E, \mathrm{spec}\widetilde{H^{\widetilde{\xi}}_{L_0}})}{2\sqrt{d(E, \mathrm{spec}\widetilde{H^{\widetilde{\xi}}_{L_0}}) + c(1 + \sup_{a \in \mathbb{Z}^2 \cap \square_{L_0 - 2}} \|\chi_a \xi\|_{C^{-1-\epsilon}(\mathbb{R}^2)} + \sup_{a \in \mathbb{Z}^2} \|\chi_a Y_{\xi, L_0 - 2}\|_{C^{-\epsilon}(\mathbb{R}^2)})}}\right)$$

$$\because \|e^{-v\cdot x}(\widetilde{H^{\widetilde{\xi}}_{L_0}} - E)^{-1} e^{v\cdot x}\|_{op} \text{ for } v \in \mathbb{R}^2 : |v| < \sqrt{d(E, \mathrm{spec}\widetilde{H^{\widetilde{\xi}}_{L_0}})}$$

$$\leq \|(\widetilde{H^{\widetilde{\xi}}_{L_0}} - |v|^2 - E)^{-1}\|_{op} \sum_{n=0}^{\infty} \|(\widetilde{H^{\widetilde{\xi}}_{L_0}} - |v|^2 - E)^{-1/2} 2v \cdot \nabla (\widetilde{H^{\widetilde{\xi}}_{L_0}} - |v|^2 - E)^{-1/2}\|_{op}$$

Initial estimate

$$\mathbb{P}(\|\chi_x(\widetilde{H^{\tilde{\xi}}_{L_0}} - E)^{-1}\chi_y\| \lesssim \exp(-m|x-y|) \text{ for } E \leq E_0) > 1 - 1/L_0^{p/2}$$

for sufficiently low $E_0 < 0$

$\Leftarrow$ inf spec $\widetilde{H^{\tilde{\xi}}_{L_0}} \geq -k_{\xi,L-2}$

k_{ξ,L_0-2} : a positive polynomial of $\|\xi_{L_0-2}\|_{C^{-1-\epsilon}(\mathbb{R}^2)}$ and $\|Y_{\xi,L_0-2}\|_{C^{-\epsilon}(\mathbb{R}^2)}$

$\mathbb{P}(\|\xi_{L_0-2}\|_{C^{-1-\epsilon}(\mathbb{R}^2)} \leq c_1\sqrt{\log(2+L_0)}$ and

$\|Y_{\xi,L_0}\|_{C^{-\epsilon}(\mathbb{R}^2)} \leq c_2\exp(-c_3 d(a,\square_{L_0}))) > 1 - 1/L_0^{p/2}$

Geometric resolvent inequality for a smooth potential V

$\square_{L_k,\boldsymbol{a}} \subset \square_{L_{k+1}}$, $a \in \square_{L_k-8,\boldsymbol{a}}$, $a_* \notin \square_{L_k,\boldsymbol{a}}$

$\phi \in C_0^\infty(\square_{L_k-2,\boldsymbol{a}} \to [0,1])$ $\phi = 1$ on $\square_{L_k-6,\boldsymbol{a}}$

$$(-\Delta + V1_{\square_{L_{k+1}}} - z)^{-1}\phi - \phi(-\Delta + V1_{\square_{L_k,\boldsymbol{a}}} - z)^{-1}$$

$$= (-\Delta + V1_{\square_{L_{k+1}}} - z)^{-1}((2\nabla\phi)\cdot\nabla + (\Delta\phi))(-\Delta + V1_{\square_{L_k,\boldsymbol{a}}} - z)^{-1}$$

$$\|\chi_{a_*}(-\Delta + V1_{\square_{L_{k+1}}} - z)^{-1}\chi_a\|_{op} = \|\chi_{a_*}(\ \ '' \ \ )\chi_a\|_{op}$$

$$\leq c_V \sum_{a_1 \in \mathbb{Z}^2 \cap \overline{\square_{L_k-2,\boldsymbol{a}} \setminus \square_{L_k-6,\boldsymbol{a}}}} \|\chi_{a_*}(-\Delta + V1_{\square_{L_{k+1}}} - z)^{-1}\chi_{a_1}\|_{op}\|\chi_{a_1}(-\Delta + V1_{\square_{L_k,\boldsymbol{a}}} - z)^{-1}\chi_a\|_{op}$$

$$\leq c_V^2 \sum_{\substack{a_1 \in \mathbb{Z}^2 \cap \overline{\square_{L_k-2,\boldsymbol{a}} \setminus \square_{L_k-6,\boldsymbol{a}}} \\ a_2 \in \mathbb{Z}^2 \cap \overline{\square_{L_k-2,\boldsymbol{a}_1} \setminus \square_{L_k-6,\boldsymbol{a}_1}}}} \|\chi_{a_*}(-\Delta + V1_{\square_{L_{k+1}}} - z)^{-1}\chi_{a_2}\|_{op}\|\chi_{a_2}(-\Delta + V1_{\square_{L_k,\boldsymbol{a}_1}} - z)^{-1}\chi_{a_1}\|_{op} \times \|\chi_{a_1}(-\Delta + V1_{\square_{L_k,\boldsymbol{a}}} - z)^{-1}\chi_a\|_{op}$$

Iterate until $a_n \doteqdot a_*$

Problems in extending Geometric resolvent inequality to our case

$$(\widetilde{H^{\widetilde{\xi}}_{L_{k+1}}} - z)^{-1}\phi - \phi(\widetilde{H^{\widetilde{\xi}}_{L_k,\boldsymbol{a}}} - z)^{-1}$$
$$= (\widetilde{H^{\widetilde{\xi}}_{L_{k+1}}} - z)^{-1}((2\nabla\phi)\cdot\nabla + (\Delta\phi)$$

$\nwarrow$ Need the paracontrolled calculus

$$+\underline{\mathbb{E}[\phi(\Pi(\Delta^{-loc}(\xi_{L_{k+1}-2} - \xi_{L_k-2,\boldsymbol{a}}), \xi_{L_{k+1}-2}) + \Pi(\Delta^{-loc}\xi_{L_k-2,\boldsymbol{a}}, \xi_{L_{k+1}-2} - \xi_{L_k-2,\boldsymbol{a}}))]})(\widetilde{H^{\widetilde{\xi}}_{L_k,\boldsymbol{a}}} - z)^{-1}$$

supp($''$) $\not\subset$ supp $\nabla\phi$

supp($''$) $\subset$ supp ϕ, $\lesssim \exp(-cd(\cdot, \mathbb{R}^2 \setminus \square_{L_k-2,\boldsymbol{a}}))$

These complicate our geometric resolvent inequality (see the next slide) but newly appeared parts have some properties of exponential decays.
Therefore we can do the multiscale analysis.

Our Geometric resolvent inequality

$$\|\chi_{a_*}(\widetilde{H^{\xi}_{L_{k+1}}} - z)^{-1}\chi_a\|_{op}$$ $\Xi(\boldsymbol{a}, L, \xi) := \sup_{a_0 \in \mathbb{Z}^2} \left(\begin{array}{c} \sum_{j=1}^{2} \left(\sum_{a_1 \in \mathbb{Z}^2 \cap \square_{L,\boldsymbol{a}}} \|\chi^2_{a_1}\xi\|_{C^{-1-\varepsilon}(\mathbb{R}^2)} \exp(-c_*|a_0 - a_1|^2) \right)^j \\ + \sum_{a_1 \in \mathbb{Z}^2} \|\chi^2_{a_1} Y_{\xi,L,\boldsymbol{a}}\|_{C^{-\varepsilon}(\mathbb{R}^2)} \exp(-c_*|a_0 - a_1|^2) \end{array} \right)$

$$\leq \sum_{a_1 \in \mathbb{Z}^2 \cap \overline{\square_{L_k-2,\boldsymbol{a}} \setminus \square_{L_k-6.\boldsymbol{a}}}} \|\chi_{a_*}(\widetilde{H^{\xi}_{L_{k+1}}} - z)^{-1}\chi_{a_1}\|_{op} c_1 \exp(-c_*|a_1 - a|)$$

$$+ \sum_{a_1 \in \mathbb{Z}^2 \cap \overline{\square_{L_k-2,\boldsymbol{a}} \setminus \square_{L_k-6,\boldsymbol{a}}}, a_2 \in \mathbb{Z}^2} \|\chi_{a_2}(\widetilde{H^{\xi}_{L_k,\boldsymbol{a}}} - z)^{-1} \ \chi_a\|_{op} c_2 \exp(-c_*(|a_1 - a_*| + |a_1 - a_2|^2))\Xi(\boldsymbol{a}, L_k - 2, \xi)^{1/2}$$

$$+ \sum_{a_1 \in \mathbb{Z}^2 \cap \overline{\square_{L_k-2,\boldsymbol{a}} \setminus \square_{L_k-6,\boldsymbol{a}}}, a_2 \in \mathbb{Z}^2} \|\chi_{a_*}(\widetilde{H^{\xi}_{L_{k+1}}} - z)^{-1}\chi_{a_1}\|_{op} \|\chi_{a_2}(\widetilde{H^{\xi}_{L_k,\boldsymbol{a}}} - z)^{-1}\chi_a\|_{op} \begin{array}{c} c_3 \exp(-c_*|a_1 - a_2|) \\ \times(|z| + \Xi(\boldsymbol{a}, L_k - 2, \xi))^{c_4} \end{array}$$

$$+ \sum_{a_1 \in \mathbb{Z}^2 \cap \overline{\square_{L_k-2,\boldsymbol{a}} \setminus \square_{L_k-6,\boldsymbol{a}}}, a_2, a_3 \in \mathbb{Z}^2} \|\chi_{a_*}(\widetilde{H^{\xi}_{L_{k+1}}} - z)^{-1}\chi_{a_2} \ \|_{op} \|\chi_{a_3}(\widetilde{H^{\xi}_{L_k,\boldsymbol{a}}} - z)^{-1}\chi_a\|_{op}$$

$$\times c_5 \exp(-c_*(|a_1 - a_2| + |a_1 - a_3|^2))(|z| + \Xi(\boldsymbol{0}, L_{k+1} - 2, \xi))^{c_6},$$

A Wegner type estimate

There exist finite positive constants c_0, c_1, c_2, c_3 such that

$$\mathbb{E}[\mathrm{Tr}[1_{[E-\eta,E+\eta]}(\widetilde{H_L^{\tilde{\xi}}})]] \leq c_1 \eta L^{c_2}$$

for any $E \leq -c_3$, $0 < \eta \leq 1 \wedge (-E/2)$ and $L \in 2\mathbb{N}$.
($c_2 = 2 \Rightarrow$ An upper bound of the density of states
Wegner (1981) discrete Anderson model)
The present estimate is enough for MSA since Fröhlich-Spencer (1983)

Idea of the proof of Wegner type estimates I

(Variation of energies)$\rightarrow$(Variation of random variables)

$\lambda_0 \in [E-\eta, E+\eta]$, $H_L^{\tilde{\xi}}\varphi_0 = \lambda_0\varphi_0$, $\|\varphi_0\|_{L^2(\mathbb{R}^2)} = 1$
$\chi_{out} \in C^\infty(\mathbb{R}^2 \setminus \Box_{L-1}), \chi_{in} \in C^\infty(\Box_{L-1/2})$ s.t. $\chi_{out}^2 + \chi_{in}^2 \equiv 1$.

IMS localization: $H_L^{\tilde{\xi}} = \chi_{out} H_L^{\tilde{\xi}} \chi_{out} + \chi_{in} H_L^{\tilde{\xi}} \chi_{in} - |\nabla \chi_{out}|^2 - |\nabla \chi_{in}|^2$
(Ismagilov-Morgan-Simon, Ref. Sigal(1982))

$$\chi_{out} H_L^{\tilde{\xi}} \chi_{out} \doteqdot \chi_{out}(-\Delta + \sum_{a \in \mathbb{Z}^2 \cap \Box_L \setminus \Box_{L-2}} \chi_a^2 \overline{\xi_a} - \mathbb{E}[\Pi(\Delta^{-loc}\xi_{L-2}, \xi_{L-2})])\chi_{out} \geq -c_2 \chi_{out}^2$$

Thus for low λ_0, it should be $1 = \underbrace{\|\chi_{out}\varphi_0\|_{L^2(\mathbb{R}^2)}^2}_{\text{small}} + \underbrace{\|\chi_{in}\varphi_0\|_{L^2(\mathbb{R}^2)}^2}_{\text{large}}$ so that

$\|\chi_{in}\varphi_0\|_{L^2(\mathbb{R}^2)}^2 \geq (-\lambda_0 - c_1)/B(\xi)$,
$(B(\xi) := c_4(1 + \|\xi_{L-2}\|_{\mathcal{B}^{-1-\epsilon/2}_{2/\epsilon,2/\epsilon}(\mathbb{R}^2)} + \|\xi_{L-2}\|^2_{\mathcal{B}^{-1-\epsilon/2}_{2/\epsilon,2/\epsilon}(\mathbb{R}^2)} + \|Y_{\xi,L-2}\|_{\mathcal{B}^{-\epsilon/2}_{2/\epsilon,2/\epsilon}(\mathbb{R}^2)})^{c_3})$
We assume $E \leq -c_1 - 2$ and $\lambda_0 \leq E + 1$. Then $\|\chi_{in}\varphi_0\|_{L^2(\mathbb{R}^2)}^2 \geq 1/B(\xi)$.

Idea of the proof of Wegner type estimates II

If $\lambda_0 \in [E-\eta, E+\eta]$ and $t \geq 2\eta B(\xi)/c_5$, then we have

$$(\varphi_0, \underbrace{(\widetilde{H_L^{\widetilde{\xi}}} + t \sum_{a \in \mathbb{Z}^2 \cap \square_L} \chi_a^2)}_{=\widetilde{H_L^{\widetilde{\xi}+t}}}\varphi_0)_{L^2(\mathbb{R}^2)} \geq E+\eta, \text{ where } \underset{\text{shift}}{\widetilde{\xi}+t} = ((\xi(x)+t)_{x\in\mathbb{R}^2}, (\overline{\xi_a}+t)_{a\in\mathbb{Z}^2})$$

$$(\varphi_0, (\widetilde{H_L^{\widetilde{\xi}-t}} - t \sum_{a \in \mathbb{Z}^2 \cap \square_L} \chi_a^2)\varphi_0)_{L^2(\mathbb{R}^2)} \leq E-\eta.$$

$$\mathbb{E}[\mathrm{Tr}[1_{[E-\eta,E+\eta]}(\widetilde{H_L^{\widetilde{\xi}}})]] = \sum_{n=1}^{\infty} \mathbb{E}[\mathrm{Tr}[1_{[E-\eta,E+\eta]}(\widetilde{H_L^{\widetilde{\xi}}})] : B(\xi) \in [n-1, n]]$$

$$\leq \sum_{n=1}^{\infty} \mathbb{E}\left[\mathrm{Tr}\left[\widetilde{1_{(-\infty,0]}}\left(\frac{1}{\eta}\left(H_L^{\widetilde{\xi}-2\eta n/c_5} - E\right)\right) - \widetilde{1_{(-\infty,0]}}\left(\frac{1}{\eta}\left(H_L^{\widetilde{\xi}+2\eta n/c_5} - E\right)\right)\right]\widetilde{\chi_{[n-1,n]}}(B(\xi)\right.$$

Idea of the proof of Wegner type estimates III

By the Cameron-Martin theorem,

$$
\begin{aligned}
=\sum_{n=1}^{\infty}\mathbb{E}\Big[\Big(&\prod_{a\in\mathbb{Z}^2\cap(\square_L\setminus\square_{L-2})}\int_{\mathbb{R}} d\overline{\xi}_a g\Big(\overline{\xi}_a+\frac{2\eta n}{c_5}\Big)\Big)\mathrm{Tr}[\widetilde{1_{(-\infty,0]}}((\widetilde{H_L^{\tilde{\xi}}}-E)/\eta)]\\
&\times\exp\Big(-\frac{2\eta n}{c_5}\int_{\square_{L-2}}\xi(x)dx-2\Big(\frac{\eta n(L-2)}{c_5}\Big)^2\Big)\widetilde{\chi_{[n-1,n]}}\Big(B\Big(\xi+\frac{2\eta n}{c_5}\Big)\Big)\\
&-\Big(\prod_{a\in\mathbb{Z}^2\cap(\square_L\setminus\square_{L-2})}\int_{\mathbb{R}} d\overline{\xi}_a g\Big(\overline{\xi}_a-\frac{2\eta n}{c_5}\Big)\Big)\mathrm{Tr}[\widetilde{1_{(-\infty,0]}}((\widetilde{H_L^{\tilde{\xi}}}-E)/\eta)]\\
&\times\exp\Big(\frac{2\eta n}{c_5}\int_{\square_{L-2}}\xi(x)dx-2\Big(\frac{\eta n(L-2)}{c_5}\Big)^2\Big)\widetilde{\chi_{[n-1,n]}}\Big(B\Big(\xi-\frac{2\eta n}{c_5}\Big)\Big)\Big]\\
\le\, & c_6L^{c_7}\eta \text{ (by Lemma on moments of numbers of negative eigenvalues)},
\end{aligned}
$$

where g is the probability density of the random variable $\overline{\xi_0}$

Variable Energy Multiscale Analysis (von Dreifus-Klein (1989))

For $1 \leq \forall p < \infty$, $0 < \forall m$ sufficiently small, $\forall E_1 < \forall E_0 < 0$ sufficiently small, $\exists \{L_k\}_k$ a strictly increasing sequence in $6\mathbb{N}$ s.t.

$\mathbb{P}($for $E_1 \leq \forall E \leq E_0$,

$$\|\chi_\cdot(\widetilde{H^{\tilde{\xi}}_{L_k,\boldsymbol{a}}} - E)^{-1}\chi_\cdot\| \wedge \|\chi_\cdot(\widetilde{H^{\tilde{\xi}}_{L_k,\boldsymbol{a}'}} - E)^{-1}\chi_\cdot\| \lesssim \exp(-m|\cdot|)) > 1 - L_k^{-p}$$

for any $\boldsymbol{a}, \boldsymbol{a}' \in \mathbb{Z}^2$ satisfying $|\boldsymbol{a} - \boldsymbol{a}'|_\infty > L_k + 2$.

cf. Fixed Energy Multiscale Analysis (Fröhlich and Spencer (1983))

$$\mathbb{P}(\|\chi_\cdot(\widetilde{H^{\tilde{\xi}}_{L_k,\boldsymbol{a}}} - E)^{-1}\chi_\cdot\| \lesssim \exp(-m|\cdot|)) > 1 - L_k^{-p}$$

Spectral averaging methods are used for the proof of the localization

For the existence of a Generalized Eigenfunction Expansion

$\nu > 1/4$ fix, $\langle x \rangle := (1+|x|^2)^{1/2}$ (Ref. Klein, Koines, Seifert, JFA(2002))

$\mathbb{E}[\mathrm{Tr}[\langle x\rangle^{-\nu} E(I : \widetilde{H^{\xi}})\langle x\rangle^{-\nu}]^p] < \infty$ for any bounded interval I and $p > 0$

$$\Leftarrow \sup_{\varepsilon\in(0,1]} \mathbb{E}\Big[\Big(\int_{\mathbb{R}^2} \frac{dx}{(1+|x|^2)^{2\nu}} \exp\Big(-\frac{t}{2}\widetilde{H^{\xi_\varepsilon}}\Big)(x,x)\Big)^m\Big] < \infty \text{ for } \begin{matrix}\forall m \in \mathbb{N} \\ \exists t > 0\end{matrix}$$

For $m = 1$,

$$\mathbb{E}\Big[\int_{\mathbb{R}^2} \frac{dx}{(1+|x|^2)^{2\nu}} \exp\Big(-\frac{t}{2}\widetilde{H^{\xi_\varepsilon}}\Big)(x,x)\Big]$$

$$= \int_{\mathbb{R}^2} \frac{dx}{(1+|x|^2)^{2\nu}} \mathbb{E}\Big[\exp\Big(\frac{1}{8}\int_0^t ds_1 \int_0^t ds_2 e^{2\varepsilon^2\Delta}(w_0^t(s_1), w_0^t(s_2))\Big)\Big]$$

$$\times \exp\Big(\frac{t}{2}\mathbb{E}[\Pi(\Delta^{-loc}\xi_\varepsilon, \xi_\varepsilon)]\Big)\frac{1}{2\pi t}$$

$$\le \frac{C}{t}\sup_{\varepsilon}\mathbb{E}\Big[\exp\Big(\iint_{0\le s_1,s_2\le t} ds_1 ds_2 (e^{\varepsilon^2\Delta}(w_0^t(s_1), w_0^t(s_2)) - \mathbb{E}[e^{\varepsilon^2\Delta}(w_0^t(s_1), w_0^t(s_2))])\Big)\Big]$$

$< \infty \because$ Matsuda (2022)

Generalized Eigenfunction Expansion

By $\mathbb{E}[\mathrm{Tr}[\langle x\rangle^{-\nu}E(I:\widetilde{H^\xi})\langle x\rangle^{-\nu}]^p]<\infty$ for any bounded interval I and $p>0$

$\mu^\xi(I):=\mathrm{Tr}[\langle x\rangle^{-\nu}E(I:\widetilde{H^\xi})\langle x\rangle^{-\nu}]\to$ Borel measure

By extending the Radon-Nikodym theorem,

$\mathbb{R}\ni\lambda\mapsto\exists Q^\xi(\lambda)\in\mathcal{I}_1(L^2(\mathbb{R}^2,dx))$: μ^ξ-locally integrable s.t. $Q^\xi(\lambda)\geq 0$

$\nwarrow$ Banach space of the trace class operators

$\langle x\rangle^{-\nu}E(I:\widetilde{H^\xi})\langle x\rangle^{-\nu}=\int_I Q^\xi(\lambda)\mu^\xi(d\lambda)$ in the sense of the Bochner integral

$P^\xi(\lambda):=\langle x\rangle^{\nu}Q^\xi(\lambda)\langle x\rangle^{\nu}$

$E(I:\widetilde{H^\xi})=\int_I P^\xi(\lambda)\mu^\xi(d\lambda)$ in $\mathcal{I}_1(L^2(\mathbb{R}^2,\langle x\rangle^{2\nu}dx),L^2(\mathbb{R}^2,\langle x\rangle^{-2\nu}dx))$

$:=\{\langle x\rangle^{\nu}A\langle x\rangle^{\nu}:A\in\mathcal{I}_1(L^2(\mathbb{R}^2,dx))\}$

with the norm $\|\langle x\rangle^{-\nu}(\cdot)\langle x\rangle^{-\nu}\|_{\mathcal{I}_1(L^2(\mathbb{R}^2,dx))}$

Generalized Eigenfunction

$\psi \in L^2(\mathbb{R}^2, \langle x \rangle^{2\nu} dx) \Rightarrow \Psi := P^\xi(\lambda)\psi \in L^2(\mathbb{R}^2, \langle x \rangle^{-2\nu} dx)$

For $\forall \varphi \in \mathrm{Dom}_{+0}(\widetilde{H^\xi}) \subset L^2(\mathbb{R}^2, \langle x \rangle^{2\nu} dx)$, we have $\widetilde{H^\xi}\varphi \in L^2(\mathbb{R}^2, \langle x \rangle^{2\nu} dx)$ and

$$\int_{\mathbb{R}^2} dx \Psi(x)(\widetilde{H^\xi}\varphi)(x) = \lambda \int_{\mathbb{R}^2} dx \Psi(x)\varphi(x)$$

$\Psi \neq 0 \Rightarrow \Psi$: a generalized eigenfunction of $\widetilde{H^\xi}$ with a generalized eigenvalue λ

As in the geometric resolvent inequality, we have

Eigenfunction Decay Inequality

$$
\begin{aligned}
&\|\chi_{\boldsymbol{a}}\Psi\|_{L^2(\mathbb{R}^2)} \\
&\leq c_3 \sum_{a_1\in\mathbb{Z}^2} \|\chi_{a_1}\Psi\|_{L^2(\mathbb{R}^2)} (1\vee \Xi(\boldsymbol{a},L-2,\xi))^{3/2}\Xi_c(a_1,\boldsymbol{a},L-2,\xi)^{1/2} \\
&\times(1\vee\Xi_c(a_1,\boldsymbol{a},L-2,\xi))^{1/2}\exp(-c_1|a_1-a_2|) \\
&+c_3\sum_{a_1\in\mathbb{Z}^2}\|\chi_{a_1}\Psi\|_{L^2(\mathbb{R}^2)}(1\vee\Xi(\boldsymbol{a},L-2,\xi))^2\exp(-c_1|a_1-a_2|-c_1 d(a_1,\square_{2L/3}(\boldsymbol{a})\setminus\square_{L/3}(\boldsymbol{a}))) \\
&+c_3\sum_{a_1,a_2\in\mathbb{Z}^2}\|\chi_{a_1}\Psi\|_{L^2(\mathbb{R}^2)}\sum_{a_3\in\mathbb{Z}^2\cap\overline{\square_2(\boldsymbol{a})}}\|\chi_{a_2}(\widetilde{H^{\tilde{\xi}}_{L,\boldsymbol{a}}}-E)^{-1}\chi_{a_3}^2\|_{op}(\log L)^{c_2} \\
&\times(1\vee\Xi(\boldsymbol{a},L-2,\xi))^2(1\vee\Xi_c(a_2,\boldsymbol{a},L-2,\xi))^{3/2}\Xi_c(a_2,\boldsymbol{a},L-2,\xi)^{1/2}(1\vee\Xi_c(a_2,\xi))^{1/2} \\
&\times(\max|\overline{\xi_0}|+|E|+\Xi(\boldsymbol{a},L-2,\xi))\exp(-c_1|a_1-a_2|) \\
&+c_3\sum_{a_1,a_2\in\mathbb{Z}^2}\|\chi_{a_1}\Psi\|_{L^2(\mathbb{R}^2)}\sum_{a_3\in\mathbb{Z}^2\cap\overline{\square_2(\boldsymbol{a})}}\|\chi_{a_2}(\widetilde{H^{\tilde{\xi}}_{L,\boldsymbol{a}}}-E)^{-1}\chi_{a_3}^2\|_{op}(\log L)^{c_2} \\
&\times(1\vee\Xi(\boldsymbol{a},L-2,\xi))^{5/2}(1\vee\Xi_c(a_2,\boldsymbol{a},L-2,\xi))(\max|\overline{\xi_0}|+|E|+\Xi(\boldsymbol{a},L-2,\xi)) \\
&\times\exp(-c_1|a_1-a_2|-c_1 d(a_1,\square_{2L/3}(\boldsymbol{a})\setminus\square_{L/3}(\boldsymbol{a})))
\end{aligned}
$$

Notations in the Eigenfunction Decay Inequality

$$\Xi_c(a_1, \boldsymbol{a}, L-2, \xi) := \sum_{a_2 \in \mathbb{Z}^2 \setminus \square_{L-2}(\boldsymbol{a})} \|\chi_{a_2}^2 \xi\|_{C^{-1-\varepsilon}(\mathbb{R}^2)} \exp(-c_1|a_1 - a_2|)$$

$$+ \sum_{(a_2,a_3) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \setminus \square_{L-2}(\boldsymbol{a})^2} \prod_{j=2}^{3} \|\chi_{a_j}^2 \xi\|_{C^{-1-\varepsilon}(\mathbb{R}^2)} \exp(-c_1|a_1 - a_j|)$$

$$+ \sum_{a_2 \in \mathbb{Z}^2} \|\chi_{a_2}^2 (Y_\xi - Y_{\xi, L-2, \boldsymbol{a}})\|_{C^{-\varepsilon}(\mathbb{R}^2)} \exp(-c_1|a_1 - a_2|),$$

$$\Xi_c(a_1, \xi) := \sum_{j=1}^{2} \left(\sum_{a_2 \in \mathbb{Z}^2} \|\chi_{a_2}^2 \xi\|_{C^{-1-\varepsilon}(\mathbb{R}^2)} \exp(-c_1|a_1 - a_2|) \right)^j$$

$$+ \sum_{a_2 \in \mathbb{Z}^2} \|\chi_{a_2}^2 Y_\xi\|_{C^{-\varepsilon}(\mathbb{R}^2)} \exp(-c_1|a_1 - a_2|)$$

Exponential localization

From the eigenfunction inequality

and the estimates of $\{\|\chi_{a_1}(\widetilde{H^{\tilde{\xi}}_{L,\boldsymbol{a}}} - E)^{-1}\chi^2_{a_2}\|_{op}\}_{a_1,a_2,L}$,
we obtain

$\|\chi_{\boldsymbol{a}}\Psi\|_{L^2(\mathbb{R}^2)} \leq c_1 \exp(-c_2|\boldsymbol{a}|_\infty)$