On the integrated density of states of random Pauli Hamiltonians

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ABSTRACT

The difference of the integrated densities of states (IDS) of the two components of a random Pauli Hamiltonian is shown to equal a constant given in terms of the expectation of the magnetic field. This formula is a random version of the Aharonov and Casher theory or that of the Atiyah and Singer index theorem. By this formula, the IDS is shown to jump at 0 if the expectation of the magnetic field is nonzero. For simple cases where the expectation of the magnetic field is zero, a lower estimate of the asymptotics of the IDS at 0 is given. This lower estimate shows that the IDS decays slower than known results for random Schrödinger operators whose infimum of the spectrum is 0. Moreover the strong-magnetic-field limit of the IDS is identified in a general setting.

1. **Introduction**

Let $B = (B_\omega(x))$ $(\omega \in \Omega, x \in \mathbb{R}^2)$ be a real random field such that

(a-i)$B_\omega(x)$ is stationary and ergodic with respect to the shift in the variable x;

(a-ii) the sample path $\mathbb{R}^2 \ni x \mapsto B_\omega(x)$ is continuous;

(a-iii) there exists $\alpha > 0$ such that $E[\exp(\alpha|B_\omega(0)|)] < \infty$;

(a-iv) there exists an $\mathbb{R}^2$-valued random field $A = (A^j_\omega(x))^2_{j=1}$ such that $\mathbb{R}^2 \ni x \mapsto A_\omega(x)$ is continuous, $\partial_1 A^2_\omega(x) - \partial_2 A^1_\omega(x) = B_\omega(x)$ and $\partial_1 A^1_\omega(x) + \partial_2 A^2_\omega(x) = 0$ in the sense of distributions.

For each ω, regarding $x \mapsto B_\omega(x)$ as a magnetic field, and $x \mapsto A_\omega(x)$ as its vector potential, we consider a Pauli Hamiltonian $H_\omega = H_\omega^+ \oplus H_\omega^-$ formally defined by

$$H_\omega^\pm = \sum_{j=1}^{2} (i\partial_j + A_\omega^j(x))^2 \pm B_\omega(x), \tag{1.1}$$

where $i = \sqrt{-1}$ and $\partial_j = \partial/(\partial x^j)$. Let $N^\pm(\lambda)$, $\lambda \in \mathbb{R}$, be the integrated density of states of the operators $H_\omega^\pm$. We define those as right continuous functions. For the exact definition, see Section 2 below.

The first result in this paper is the following:

Theorem 1. *Under the conditions* (a-i)–(a-iv)*, we have*

$$N^-(\lambda) - N^+(\lambda) = E[B_\omega(0)]/(2\pi) \tag{1.2}$$

for any $\lambda \geq 0$.

This is an extension of the index theorem to a noncompact setting in terms of the integrated density of states. As in the usual index theorem, a main tool for the proof is the theory on the supersymmetry (cf. Section 6.3 in [18]): by the supersymmetry we have

$$N^-(\lambda) - N^+(\lambda) = \widetilde{N^-}(t) - \widetilde{N^+}(t) \tag{1.3}$$

for any $\lambda \geq 0$ and $t > 0$, where $\widetilde{N^\pm}(t)$ is the Laplace-Stieltjes transform of $N^\pm(\lambda)$. Then (1.2) is obtained by identifying the limit of the right hand side of (1.3) represented in terms of the heat semigroup as $t \downarrow 0$. This is the same technique for the proof of the index theorem (cf. [4], [18], [33], [43]). However in our case, we should prove that the boundary condition to define the integrated density of states does not affect the result. We carry out this in a general setting and this is our main contribution. For this, see the proof of Lemma 4.1 and Remark 4.1 below.

From this theorem, we easily know on the kernels as follows:

Corollary 1. *Under the conditions* (a-i)–(a-iv), *we have*

$$N^{\pm}(0) \geq E[B_\omega(0)]_{\mp}/(2\pi), \tag{1.4}$$

where $a_\pm = \max\{\pm a, 0\}$ *for any* $a \in \mathbb{R}$. *Therefore, if* $\mp E[B_\omega(0)] > 0$, *then* 0 *is the eigenvalue of* $H_\omega^\pm$ *with infinite multiplicity.*

Corollary 2. *Under the conditions* (a-i)–(a-iv) *and* $H_\omega^+ \geq b$ *for some* $b \in (0, \infty)$, *we have*

$$N^-(0) = E[B_\omega(0)]/(2\pi) \tag{1.5}$$

and

$$\inf\{\lambda : N^+(\lambda) \neq 0\} = \inf\{\lambda : N^-(\lambda) - N^-(0) \neq 0\} \geq b. \tag{1.6}$$

Example 1.1. (i) If $b_1 := \text{ess sup}_{\omega\in\Omega} \inf_{x\in\mathbb{R}^2} B_\omega(x) > 0$, then we have (1.5) and (1.6) with $b = 2b_1$ by the well known estimate $H_\omega^+ \geq 2B_\omega(x)$ (cf. Avron, Herbst and Simon [7] Theorem 2.9).

(ii) If there exist a random variable B_ω^0 and a random field $\varphi_\omega(x)$ such that $\varphi_\omega(x)$ is C^2 in $x \in \mathbb{R}^2$, $B_\omega(x) = B_\omega^0 + \Delta\varphi_\omega(x)$ and $b_2 := \text{ess sup}_{\omega\in\Omega} B_\omega^0 \exp(-2\text{osc }\varphi_\omega) > 0$, then we have (1.5) and (1.6) with $b = 2b_2$ by a Raikov's result (Proposition 1.2 in [50]), where Δ is the Laplacian and $\text{osc }\varphi_\omega = \sup_{x\in\mathbb{R}^2}\varphi_\omega(x) - \inf_{x\in\mathbb{R}^2}\varphi_\omega(x)$. If the first derivatives of $\varphi_\omega(x)$ are almost surely bounded on $\mathbb{R}^2$, then B_ω^0 is not random and coincides with the expectation $E[B_\omega(0)]$ by the ergodic theorem and Green's formula.

These results are extended to arbitrary even dimensional space as Theorem 4 below. In this paper we prove the general theorem.

On the dimension of the kernel of the deterministic Pauli Hamiltonian $H_0 = H_0^+ \oplus H_0^-$ obtained by replacing $(A_\omega^j(x))_{j=1}^2$ and $B_\omega(x)$ in (1.1) by a deterministic vector potential

$(A_0^j(x))_{j=1}^2$ and its magnetic field $B_0(x)$, respectively, we have Aharonov and Casher's theory [2]. Their theory gave a basis constituting

$$\left\lfloor \frac{1}{2\pi}\int_{\mathbb{R}^2} B_0(x)dx \right\rfloor$$

elements of the kernel of H_0^- under the compactness of $\mathrm{supp} B_0$ and

$$1 \le \frac{1}{2\pi}\int B_0(x)dx < \infty,$$

where $\lfloor a \rfloor$ is the largest integer smaller than a for any $a > 0$ (cf.[10], [18]). Their theory has been extended to many situations where $B_0(x)$ is integrable in x. For this aspect, see [8], [10], [18], [24], [25] and the references therein. For the case that $B_0(x)$ is periodic as $B_0(x^1 + T^1, x^2) = B_0(x^1, x^2 + T^2) = B_0(x)$, Dubrovin and Novikov [21], [22] gave a Bloch basis and showed the existence of the gap between zero and the rest of the spectrum under the condition that

$$\frac{1}{2\pi}\int_0^{T^1}\int_0^{T^2} B_0(x)dx^1dx^2 \tag{1.7}$$

is a natural number. From their result, we see that $N_0^-(0)$ equals the natural number in (1.7) and $\inf\{\lambda : N_0^-(\lambda) \ne N_0^-(0)\} > 0$, where $N_0^-(\lambda)$ is the integrated density of states of H_0^- (cf. [29]). For the case that

$$B_0(x) = B_0 + \widetilde{B_0}(x) \tag{1.8}$$

is almost periodic such that $B_0 > 0$,

$$\widetilde{B_0}(x) = \mathrm{Re}\sum_{n=1}^{\infty} B_n e^{iC_n x},$$

and

$$\sum_{n=1}^{\infty} |B_n|(|C_n|^{-2} + 1) < \infty,$$

Raikov [50] showed the same formulas as (1.5) and (1.6) where $N^{\pm}$, $E[B_\omega(0)]$ and b are replaced by $N_0^{\pm}$, B_0 and $2B_0\exp(-2\mathrm{osc}\,\varphi)$, respectively, where

$$\varphi(x) = \Delta^{-1}\widetilde{B_0}(x) = \mathrm{Re}\sum_{n=1}^{\infty}\frac{-B_n}{C_n^2}e^{iC_n x}.$$

Since

$$\lim_{R\to\infty}\frac{1}{R^2}\left\lfloor\left|\frac{1}{2\pi}\int_{\Lambda(R)}B_\omega(x)dx\right|\right\rfloor = \frac{E[B_\omega(0)]}{2\pi}$$

for an ergodic random field $B_\omega(x)$ and

$$\lim_{R\to\infty}\frac{1}{R^2}\int_{\Lambda(R)}B_0(x)dx = B_0$$

for a function $B_0(x)$ given by (1.8), our result is regarded as an extension of [2], [21], [22] and [50] to a general stationary ergodic magnetic field.

We next consider a strong-magnetic-field asymptotics: we consider the asymptotics as $\xi\to\infty$ of the integrated density of states $N^{\pm}(\lambda;\xi)$ of the operator formally defined by

$$H_\omega^{\xi,\pm} = \sum_{j=1}^{2}(i\partial_j + \xi A_\omega^j(x))^2 \pm \xi B_\omega(x). \tag{1.9}$$

We assume

(a-v) there are $c>0$ and $\nu>1$ such that $E[\exp(\alpha|B_\omega(0)|)] \le c\exp(\alpha^\nu)$ for any $\alpha>0$.

Then by the localizing effect of the strong magnetic field, we can determine the leading term:

Theorem 2. *Under the conditions* (a-i), (a-ii), (a-iv) *and* (a-v), *the integrated density of states $N^{\pm}(\lambda;\xi)$ of the operators $H_\omega^{\xi,\pm}$ satisfies*

$$\lim_{\xi\to\infty}\frac{N^{\pm}(\lambda;\xi)}{\xi} = \frac{E[B_\omega(0)_\mp]}{2\pi} \tag{1.10}$$

for each $\lambda>0$.

In the right hand side of (1.10), the positive or negative part is taken inside the expectation whereas that is taken outside of the expectation in (1.4). This is due to the localizing effect of the strong magnetic field. The right hand side of (1.10) may be greater than that of (1.4). However we cannot estimate the value of $N^{\pm}(0)$ by (1.10), since the equation (1.10) is not proven uniformly on an interval including 0.

The strong-magnetic-field asymptotics of the integrated density of states has been extensively studied for the Pauli Hamiltonian with the deterministic constant magnetic field perturbed by a random electric scalar potential [11], [38], [39], [41], [49], [54], [55], [56]. For variable magnetic fields, we have related deterministic results [23], [48]. In [48], Raikov gives the leading term of the counting function of the number of the eigenvalues. Moreover in [38], [48] and [49], the three dimensional cases are also treated, where the degeneracy of the magnetic field brings another problem.

In this paper we concentrate only on the unperturbed Pauli Hamiltonian with a non-degenerate magnetic field and prove the fundamental result in a considerably general situation: we prove a generalization of Theorem 2 to arbitrary even dimensional space. The result is Theorem 5 below.

To prove the upper estimate, we use a method of Erdös [23] proving

$$\lim_{\xi\to\infty}\frac{1}{\xi}E(\{0\};0,0;\xi)=\frac{|B_0(0)|}{2\pi} \tag{1.11}$$

under some conditions, where $E(\Lambda;x,y;\xi)$, $\Lambda\in\mathcal{B}(\mathbb{R})$, x, $y\in\mathbb{R}^2$, $\xi>0$, is the integral kernel of the resolution of the identity of the deterministic Pauli Hamiltonian defined by $H_0^{\xi}=H_0^{\xi,+}\oplus H_0^{\xi,-}$ obtained by replacing $(A_{\omega}^j(x))_{j=1}^2$ and $B_{\omega}(x)$ in (1.9) by a deterministic vector potential $(A_0^j(x))_{j=1}^2$ and its magnetic field $B_0(x)$, respectively. By the same proof, we can also show

$$\lim_{\xi\to\infty}\frac{1}{\xi}E^{\pm}([0,\lambda];0,0;\xi)=\frac{B_0(0)_{\mp}}{2\pi}$$

for each $\lambda>0$, where $E_{\omega}^{\pm}(\Lambda;x,y;\xi)$ is the integral kernel of the resolution of the identity of $H_0^{\xi,\pm}$. This result relates with (1.10) by the representation $N^{\pm}(\lambda;\xi)=E[E_{\omega}^{\pm}([0,\lambda];0,0;\xi)]$, where $E_{\omega}^{\pm}(\Lambda;x,y;\xi)$ is the integral kernel of the resolution of the identity of the random

Pauli Hamiltonian $H_\omega^{\xi,\pm}$ (cf. [13]). To show the upper estimate for (1.11), he determined the limit of a function defined by using the corresponding integral kernel of the heat semigroup represented by the Feynman-Kac-Itô formula. Since the Laplace-Stieltjes transform of $N^\pm(\lambda;\xi)$ also has a similar representation (see Lemma 3.1 below), we can show the upper estimate by the same method.

To prove the lower estimate, we estimate the number of low lying eigenvalues of the Hamiltonian restricted to small domains. For this we estimate the energy of the functions obtained by restricting the functions in Aharonov and Casher's theory to small domains. For the higher dimensional case, we need extra work to show the lower estimate: to construct the functions in Aharonov and Casher's theory, we use the theory of the $\overline{\partial}$-Neumann problem [9]. For this we assume that the magnetic field is derived from a skew-Hermitian matrix valued random field. Moreover our estimate of the number of low lying eigenvalues of the restricted Hamiltonian becomes best when the domain is a polydisc depending on ω. Then, to obtain a sharp estimate of $N(\lambda;\xi)$, we should take the osculatory packing so that the related quantities are measurable.

We next consider the low energy asymptotics of $N^\pm(\lambda)$. For this, Casher and Neuberger [15] showed

$$N^\pm(\lambda) \sim c\sqrt{\lambda} \qquad \text{as } \lambda \downarrow 0, \tag{1.12}$$

when $B_\omega(x)$ is a Gaussian white noise with mean zero, by a heuristic argument using Aharonov and Casher's theory and a techniques in the field theory. The equation (1.12) means that (1.5) may hold without the strict positivity of H_ω^+ and that (1.10) may not hold at $\lambda = 0$. More interesting point of (1.12) is that the decay of $N^\pm(\lambda)$ as $\lambda \downarrow 0$ is slower than that of the free Hamiltonian $-\Delta$. This behavior is in contrary to those of other random Schrödinger operators (cf.[14], [46]). The same type of behavior was also shown by Comtet, Georges and Le Doussal [16], [17], where

$$N^{\pm}(\lambda) \sim c(\log(1/\lambda))^{-3} \qquad \text{as } \lambda \downarrow 0. \tag{1.13}$$

is shown when $A_\omega^1(x)$ is a Gaussian white noise depending only on x^1 and $A_\omega^2 = 0$ by a heuristic argument reducing the problem to a 1-dimensional problem and representing $N^{\pm}(\lambda)$ in terms of Bessel functions. More recently, Ludwig, Fisher, Shankar and Grinstein [40], Motrunich, Damle and Huse [44] and Fukui [27] investigated the asymptotics of the density of states of the Dirac operator by various heuristic arguments when A_ω^1 and A_ω^2 are independent Gaussian white noises. In this case, their theory implies

$$N^{\pm}(\lambda) \sim c\lambda^{\alpha} \qquad \text{as } \lambda \downarrow 0, \tag{1.14}$$

where α is a constant in the interval $(0, 1)$ determined by the covariance of the white noise. The same behavior was also shown by a heuristic argument in Horovitz and Le Doussal [30] when $A_\omega(x) = (-\partial_2 C_\omega(x), \partial_1 C_\omega(x))$ and $C_\omega(x)$ is a Gaussian random field such that $E[(C_\omega(x) - C_\omega(x'))^2] \sim \log|x - x'|$ as $|x - x'| \to \infty$.

In this paper, we give a rigorous lower estimate proven by slightly modifying the argument used in the lower estimate for (1.10). We consider simple cases as follows:

(a-vi) $B_\omega(x) = B_\omega^1(x^1) + B_\omega^2(x^2)$, where $B_\omega^1(x^1)$ and $B_\omega^2(x^2)$ are independent Gaussian random processes with the mean zero and the covariance $\beta^j(x^j) = E[B_\omega^j(x^j)B_\omega^j(0)]$, $j = 1, 2$ satisfying $0 \le \beta^j(x^j) \in L^1(\mathbb{R})$ and $\inf_{|x^j| \le r} \beta^j(x^j) > 0$ for some $r > 0$.

(a-vii) $B_\omega(x) = B_\omega^1(x^1)$, where $B_\omega^1(x^1)$ is a Gaussian random process with the mean zero and the covariance $\beta^1(x^1) = E[B_\omega^1(x^1)B_\omega^1(0)]$ satisfying $0 \le \beta^1(x^1) \in L^1(\mathbb{R})$ and $\inf_{|x^1| \le r} \beta^1(x^1) > 0$ for some $r > 0$.

In these cases, we have the following:

Theorem 3. *Under the conditions* (a-i), (a-ii) *and either* (a-vi) *or* (a-vii), *the integrated density of states* $N^{\pm}(\lambda)$ *of the operators* $H_\omega^{\pm}$ *satisfy*

$$\liminf_{\lambda \downarrow 0} N^{\pm}(\lambda)(\log(1/\lambda))^{1/3} > 0. \tag{1.15}$$

This theorem indicates that our integrated density of states decays slower than those in any known results for random Schrödinger operators whose infimum of the spectrum is 0. We prove this theorem in Section 6 below.

The organization of this paper is as follows. In the next section, we formulate our problem in a general setting and state the general theorems, Theorem 4 and Theorem 5. In Section 3, we first recall a representation of the integrated density of states by the Feynman-Kac-Itô formula in [53] to prove an upper estimate for Theorem 5. The same representation is used to prove Theorem 4 by using also a supersymmetry in Section 4. After that, in Section 5, we prove a lower estimate for Theorem 5 by referring Aharonov and Casher's theory. The same techniques are used in Section 6 to prove Theorem 3.

2. **A general setting**

Let $d = 2h$ be a positive even number. Let $B = (B^j_{\omega,k}(x))_{1 \le j,k \le d}$ $(\omega \in \Omega, x \in \mathbb{R}^d)$ be a real skew-symmetric matrix valued random field such that

(b-i) B is stationary and ergodic with respect to the shift in the variable x;

(b-ii) the sample path $x \mapsto B_\omega(x)$ is continuous;

(b-iii) the corresponding form is closed: $d\{\sum_{j<k} B^j_{\omega,k}(x) dx^j \wedge dx^k\} = 0$ in the sense of distributions;

(b-iv) there exists $\alpha > 0$ such that $E[\exp(\alpha|B^j_{\omega,k}(0)|)] < \infty$ for any $1 \le j < k \le d$;

We give a vector potential $A_\omega = (A^j_\omega(x))_{1 \le j \le d}$ by the Poincaré gauge:

$$A^j_\omega(x) := \int_0^1 \sum_{k=1}^d B^k_{\omega,j}(tx) t x^k dt. \tag{2.1}$$

This vector potential is continuous in x and satisfies

$$d\Big\{\sum_{j=1}^{d} A^j_\omega(x)dx^j\Big\} = \sum_{j<k} B^j_{\omega,k}(x)dx^j \wedge dx^k$$

in the sense of distributions. Let $\gamma_1, \gamma_2, \dots, \gamma_d$ be Hermitian matrices acting a 2^h-dimensional complex Hilbert space V and satisfying the commutation relation

$$\gamma_j\gamma_k + \gamma_k\gamma_j = \begin{cases} 2I & \text{if } j = k, \\ O & \text{if } j \neq k, \end{cases} \tag{2.2}$$

where I and O are the identity and the zero matrices, respectively (see e.g. [18] §12.2). Then we can define the random Dirac operator by

$$D_\omega := \sum_{j=1}^{d} \gamma_j(i\partial_j + A^j_\omega(x))$$

acting on the space $C_0^\infty(\mathbb{R}^d \to V)$ of V-valued smooth functions with compact supports. It is known that this operator is essentially self-adjoint in the space $L^2(\mathbb{R}^d \to V)$ of V-valued L^2-functions (cf. [35], [34] §9.2.1). We denote the unique self-adjoint extension by the same symbol. The random Pauli Hamiltonian is the self-adjoint operator defined by $H_\omega = D_\omega^2$.

We use an Hermitian matrix $\Gamma = i^h\gamma_1\gamma_2\cdots\gamma_d$, which depends on the orientation of $\mathbb{R}^d$. The eigenvalues of this matrix are 1 and -1. The projection to the eigenspace $V_\pm = \{\Phi \in V : \Gamma\Phi = \pm\Phi\}$ is $\Gamma_\pm = (I \pm \Gamma)/2$. Since $D_\omega\Gamma = -\Gamma D_\omega$, the operator H_ω is regarded as the direct sum of the two self-adjoint operators H_ω^+ and H_ω^- on the space $L^2(\mathbb{R}^d \to V_+)$ and $L^2(\mathbb{R}^d \to V_-)$, respectively.

We define the integrated density of states $N^\pm(\lambda)$, $\lambda \in \mathbb{R}$, of the random Pauli Hamiltonian $H_\omega^\pm$ by the usual method: for each $R > 0$, let $\Lambda(R) := (-R/2, R/2)^d$. Let $H_\omega^{\pm,R}$ be the self-adjoint operator on the space $L^2(\Lambda(R) \to V_\pm)$ with the Dirichlet boundary condition: $H_\omega^{\pm,R}$ is the self-adjoint operator corresponding to the closure of the quadratic form $q_\omega(\Phi, \Psi) := \langle D_\omega\Phi(x), D_\omega\Psi(x)\rangle_V$ with the domain $C_0^\infty(\Lambda(R) \to V_\pm)$, where $\langle\cdot,\cdot\rangle_V$ is the Hermitian inner product of the space $L^2(\mathbb{R}^d \to V)$. Then the spectra of $H_\omega^{\pm,R}$ are purely discrete. Let $N_\omega^{\pm,R}(\lambda)$ be its counting function, which is the number of eigenvalues of $H_\omega^{\pm,R}(\lambda)$ not exceeding λ. Then, by the method by Kirsch-Martinelli (cf. [14], [37]), we can show the existence of a deterministic, right continuous, increasing function $N^\pm(\lambda)$ such that

$$R^{-d} N_\omega^{\pm,R}(\lambda) \longrightarrow N^\pm(\lambda) \tag{2.3}$$

as $R \to \infty$ for any point of continuity of $N^\pm(\lambda)$ and almost all ω. We here remark that the method by Kirsch-Martinelli cannot be applied directly since the vector potential $A_\omega(x)$ appearing in the Hamiltonian is not stationary. However this difficulty can be overcome by using the gauge invariance as in Lemma 3.1 of [53].

Remark 2.1. (i) As is stated in Remark 2.1 in [53], we can use other boundary conditions to define the integrated density of states: if we define an operator $\widehat{H}_\omega^{\pm,R}$ with a Neumann boundary condition by the corresponding self-adjoint operator to the quadratic form

$$\begin{aligned} \widehat{q}_\omega(\Phi,\Psi) := & \sum_{j=1}^d \langle (i\partial_j + A_\omega^j(x))\Phi(x), (i\partial_j + A_\omega^j(x))\Psi(x)\rangle_V \\ & + \langle \Phi(x), \sum_{j<k} i\gamma_j\gamma_k B_{\omega,k}^j(x)\Psi(x)\rangle_V \end{aligned} \tag{2.4}$$

with the domain

$$\begin{aligned} & W^{1,2}(\Lambda(R) \to V_\pm) \\ & := \{\Phi \in L^2(\Lambda(R) \to V_\pm) : \partial_j \Phi \in L^2(\Lambda(R) \to V_\pm) \text{ for } j = 1, 2, \ldots, d\} \end{aligned}$$

and denote the corresponding counting function of the spectra by $\widehat{N}_\omega^{\pm,R}(\lambda)$, then we have

$$R^{-d} \widehat{N}_\omega^{\pm,R}(\lambda) \longrightarrow N^\pm(\lambda) \tag{2.5}$$

as $R \to \infty$ for any point of continuity of $N^\pm(\lambda)$ and almost all ω. For the details, refer [19], [31], [32] and [45].

(ii) To define the integrated density of states by a Neumann boundary condition, it is important to use the form $\widehat{q}_\omega$ instead of q_ω, since the spectrum of the self-adjoint operator $\widetilde{H}_\omega^{\pm,R}$ corresponding to the quadratic form q_ω with the domain

$$Q := \{\Phi \in L^2(\Lambda(R) \to V_\pm) : \sum_{j=1}^{d} \gamma_j \partial_j \Phi \in L^2(\Lambda(R) \to V_\pm)\}$$

may not be discrete. In fact, when $d = 2$ and the magnetic field is a positive constant B, the Dirac operator and the above domain are represented as

$$D = -2i \begin{pmatrix} 0 & \partial/(\partial z) - B\overline{z}/4 \\ \partial/(\partial \overline{z}) - Bz/4 & 0 \end{pmatrix}$$

and

$$Q = \{\Phi = {}^t(\Phi_1, \Phi_2) : \Phi_1, \Phi_2, \partial\Phi_1/(\partial\overline{z}), \partial\Phi_2/(\partial z) \in L^2(\Lambda(R))\},$$

respectively, in terms of the complex coordinate $z = x^1 + ix^2$ by the representation of $\{\gamma_j\}_j$ in (2.12) below. Then the functions

$$\Phi_n = {}^t(z^n \exp(-B|z|^2/4), 0), \;\; n \in \mathbb{N},$$

satisfy $D\Phi_n = 0$ and belong to Q. Thus they belong to the kernel of the Hamiltonian. Moreover they are linearly independent by the uniqueness theorem in the complex analysis. Therefore the dimension of the kernel of the Hamiltonian is infinity. This fact does not contradict a general theory on the discreteness of the spectrum of an elliptic differential operator on a bounded domain (cf. [1] Theorem 14.6). In fact the domain of the Hamiltonian does not satisfy the conditions in Theorem 14.6 in [1].

For any real skewsymmetric matrix $B = (B^j_k)_{1\le j,k\le d}$, the Pffafian, $\mathrm{Pff}(B)$, is defined by

$$\mathrm{Pff}(B) = \frac{1}{2^h h!} \sum_{\sigma \in \mathfrak{S}(d)} (\mathrm{sgn}\sigma) B^{\sigma(1)}_{\sigma(2)} B^{\sigma(3)}_{\sigma(4)} \cdots B^{\sigma(d-1)}_{\sigma(d)},$$

where $\mathfrak{S}(d)$ is the set of all permutation of $\{1, 2, \ldots, d\}$ and $\mathrm{sgn}\sigma$ is the signature for each permutation σ (cf. [18] Definition 12.15).

Then Theorem 1 is extended as follows:

Theorem 4. *Under the conditions* (b-i)–(b-iv), *let* $N^{\pm}(\lambda)$ *be the integrated density of states of the random Pauli Hamiltonian* $H^{\pm}_{\omega}$. *Then it holds that*

$$N^{+}(\lambda) - N^{-}(\lambda) = \left(\frac{-1}{2\pi}\right)^{h} E[\mathrm{Pff}(B_\omega(0))] \tag{2.6}$$

for each $\lambda \geq 0$.

We prove this theorem in Section 4.

We next state a generalization of Theorem 2. For this we introduce the following conditions:

(b-v) there exist $c > 0$ and $\nu > 1$ such that $E[\exp(\alpha|B^{j}_{\omega,k}(0)|)] \leq c\exp(\alpha^{\nu})$ for any $\alpha > 0$ and $1 \leq j < k \leq d$;

(b-vi) $P(\mathrm{rank}B_\omega(0) = d) > 0$;

(b-vii) there exist a complex structure valued random variable $J_\omega = (J^{j}_{\omega,k})_{1\leq j,k\leq d}$ and $R_0 > 0$ such that $J_\omega B_\omega(x) = B_\omega(x)J_\omega$ for any $x \in \Lambda(R_0)$ and the Hermitian matrix $B_\omega(0)/i$ is nonnegative definite on the complex vector space $(\mathbb{C}^d)^{(1,0)}_{\omega} := \{v \in \mathbb{C}^d : J_\omega v = iv\}$.

We use the condition (b-vii) only to prove the lower estimate for the strong-magnetic-field asymptotics: in Lemma 5.1 below, we use the theory of the Dolbeault complex to solve the Dirac equation locally. This condition is equivalent with the following:

(b-vii)$'$ there exist a $d \times d$ orthogonal matrix valued random variable $U_\omega = (U^{j}_{\omega,k})_{1\leq j,k\leq d}$ and $R_0 > 0$ such that the skew-symmetric matrix valued random field $B'_\omega(x) = (B'^{,j}_{\omega,k}(x))_{1\leq j,k\leq d}$ defined by $B'_\omega(x) = U^{*}_{\omega}B_\omega(x)U_\omega$ satisfies

$$B'^{,2\ell-1}_{\omega,2m-1}(x) = B'^{,2\ell}_{\omega,2m}(x) \quad \text{and} \quad B'^{,2\ell-1}_{\omega,2m}(x) = -B'^{,2\ell}_{\omega,2m-1}(x) \tag{2.7}$$

for any $1 \leq \ell \leq m \leq h$ and $x \in \Lambda(R_0)$, and becomes the direct sum of $B_\omega^{(\ell)} J$, $\ell = 1, 2, \ldots, h$, at $x = 0$, where $B_\omega^{(\ell)} \geq 0$ and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.8}$$

(2.7) means that $B'_\omega(x)$ is a representation by a real matrix of a skew-Hermitian matrix $\mathbb{B}_\omega(x) = (\mathbb{B}^\ell_{\omega,m}(x))_{1 \leq \ell, m \leq h}$, where $\mathrm{Re}\mathbb{B}^\ell_{\omega,m}(x) = B'^{,2\ell-1}_{\omega,2m-1}(x) = B'^{,2\ell}_{\omega,2m}(x)$ and $\mathrm{Im}\mathbb{B}^\ell_{\omega,\ell}(x) = B'^{,2\ell-1}_{\omega,2m}(x) = -B'^{,2\ell}_{\omega,2m-1}(x)$ for any $1 \leq \ell, m \leq h$. We use the condition (b-vii) in the form of (b-vii)$'$. This condition is always satisfied when $d = 2$. For the higher dimensional case, we give the following example:

Example 2.1. For a 4×4 real skew symmetric matrix valued random field $B_\omega(x)$ to be commutative with the matrix

$$J_2 = \begin{pmatrix} J & O \\ O & J \end{pmatrix},$$

it is necessary and sufficient that

$$B_\omega(x) = \begin{pmatrix} a_\omega(x) J & c_\omega(x) E + d_\omega(x) J \\ -c_\omega(x) E + d_\omega(x) J & b_\omega(x) J \end{pmatrix},$$

where E is the 2×2 identity matrix. For this random field to satisfy the condition (b-vii) with the complex structure J_2, it is necessary and sufficient that

$$a_\omega(0) b_\omega(0) \geq c_\omega(0)^2 + d_\omega(0)^2. \tag{2.9}$$

We define D^ξ_ω, $H^{\pm,\xi}_\omega$, $H^{\pm,R,\xi}_\omega$, q^ξ_ω, $N^{\pm,R}_\omega(\lambda;\xi)$ and $N^\pm(\lambda;\xi)$ by replacing the vector potential A_ω by ξA_ω in the definitions of D_ω, $H^\pm_\omega$, $H^{\pm,R}_\omega$, q_ω, $N^{\pm,R}_\omega(\lambda)$ and $N^\pm(\lambda)$, respectively.

Then Theorem 2 is extended as follows:

Theorem 5. *Under the conditions* (b-i)–(b-iii), (b-v)–(b-vii), *let* $N^{\pm}(\lambda;\xi)$ *be the integrated density of states of the random Pauli Hamiltonian* $H_\omega^{\pm,\xi}$. *Then it holds that*

$$\lim_{\xi\uparrow\infty}\frac{N^{\pm}(\lambda;\xi)}{\xi^h}=E\left[\left\{\left(\frac{-1}{2\pi}\right)^h \mathrm{Pff}(B_\omega(0))\right\}_{\pm}\right] \tag{2.10}$$

for each $\lambda>0$.

Since $\mathrm{Pff}(B)^2=\det(B)$, to prove Theorem 5, we have only to prove

$$\lim_{\xi\uparrow\infty}\frac{N(\lambda;\xi)}{\xi^h}=\frac{E[\sqrt{\det B_\omega(0)}]}{(2\pi)^h}, \tag{2.11}$$

where $N(\lambda;\xi)=N^{+}(\lambda;\xi)+N^{-}(\lambda;\xi)$ is the integrated density of states of the Pauli Hamiltonian $H_\omega^{\xi}=H_\omega^{+,\xi}\oplus H_\omega^{-,\xi}$.

We prove the upper estimate in the next section and the lower estimate in Section 5.

In Section 1, we use the following representation of the matrices $\{\gamma_j\}_j$:

$$\gamma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\quad\text{and}\quad\gamma_2=\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix} \tag{2.12}$$

on $V=\mathbb{C}^2$. Then we have

$$D_\omega=\begin{pmatrix}0 & \mathcal{A}_\omega\\ \mathcal{A}_\omega^* & 0\end{pmatrix},$$

where

$$\mathcal{A}_\omega=(i\partial_1+A_1^\omega(x))+i(i\partial_2+A_2^\omega(x)) \tag{2.13}$$

and $\mathcal{A}_\omega^*$ is its formal adjoint.

Remark 2.2. We can replace the condition (b-i) by

(b-i)$'$ B is $\mathbb{Z}^d$-stationary and $\mathbb{Z}^d$-ergodic with respect to the shift in the variable x.

In this condition, the random field B is called $\mathbb{Z}^d$-stationary if there exist measure preserving transformations $\{T_a\}_{a\in\mathbb{Z}^d}$ on the underlying probability space Ω such as $B_{T_a\omega}(x)=$

$B_\omega(x+a)$ for all $x \in \mathbb{R}^d$ and $a \in \mathbb{Z}^d$. The $\mathbb{Z}^d$-stationary random field B is called $\mathbb{Z}^d$-ergodic if the transformations $\{T_a\}_{a\in\mathbb{Z}^d}$ are ergodic (cf. [36]).

Under this condition, we replace the conditions (b-iv)–(b-vii) appropriately. Then the integrated density of states $N^\pm(\lambda)$ or $N^\pm(\lambda;\xi)$, $\lambda \in \mathbb{R}$, of the random Pauli Hamiltonian $H^\pm_\omega$ or $H^{\pm,\xi}_\omega$ is defined similarly, Theorem 4 is modified as

$$N^+(\lambda) - N^-(\lambda) = \Big(\frac{-1}{2\pi}\Big)^h \int_{\Lambda(1)} E[\mathrm{Pff}(B_\omega(x))]dx \tag{2.14}$$

for each $\lambda \geq 0$, and Theorem 5 is modified as

$$\lim_{\xi\uparrow\infty} \frac{N^\pm(\lambda;\xi)}{\xi^h} = \int_{\Lambda(1)} E\Big[\Big\{\Big(\frac{-1}{2\pi}\Big)^h \mathrm{Pff}(B_\omega(x))\Big\}_\pm\Big]dx \tag{2.15}$$

for each $\lambda > 0$.

The proof is reduced to that for the case of the condition (b-i) as follows. We use a probability space ${}_z\Omega$ defined by the product of the original probability space Ω and the closed box $\overline{\Lambda(1)}$ with the Lebesgue measure. On this space, we define a random fields ${}_zB$ on $\mathbb{R}^d$ by ${}_zB_{(\omega,y)}(x) = B_\omega(x+y)$ for $(\omega,y) \in {}_z\Omega$ and $x \in \mathbb{R}^d$. Then it is easy to see that this random field satisfies all conditions required in Theorems 4 and 5 except for the ergodicity in the condition (b-i). However in the following proof of Theorems 4 and 5, the ergodicity is used only in Lemma 3.1 below. That lemma is extended to the present situation by slightly modifying its proof.

3. **Proof of Theorem 5: (I) Upper estimate**

In this section we assume (b-i)–(b-iii) and (b-v)–(b-vi). As in [53], we use Lemma 3.1 below: for each $R > 0$, we define a vector potential $A^R = (A^{R,j}_\omega(x))_{1\leq j\leq d}$ by

$$A^{R,j}_\omega(x) = \frac{-\Gamma(h)}{2\pi^h} \int \sum_{k\neq j} \frac{x^k - y^k}{|x-y|^d} B^{R,j}_{\omega,k}(y)dy, \tag{3.1}$$

where $\Gamma(\cdot)$ is the Gamma function,

$$B^{R,j}_{\omega,k}(y) = B^j_{\omega,k}(y)\rho\Big(\frac{y}{R}\Big) + \frac{1}{R}\Big((\partial_j\rho)\Big(\frac{y}{R}\Big)A^k_\omega(y) - (\partial_k\rho)\Big(\frac{y}{R}\Big)A^j_\omega(y)\Big)$$

and $\rho \in C_0^\infty(\Lambda(2) \to [0,1])$ such that $\rho = 1$ on $\Lambda(1)$. Let $\widetilde{H_\omega^{R,\xi}}$ be the operator obtained by replacing A by A^R in the definition of $H_\omega^{R,\xi} = H_\omega^{+,R,\xi} \oplus H_\omega^{-,R,\xi}$. Then $\widetilde{H_\omega^{R,\xi}}$ is unitarily equivalent with $H_\omega^{R,\xi}$ and satisfies the following:

Lemma 3.1. *Let*

$$\widetilde{N}\Big(\frac{t}{2};\xi\Big) := \int_0^\infty e^{-t\lambda/2} dN(\lambda;\xi)$$

and

$$\widetilde{N}^R\Big(\frac{t}{2};\xi\Big) := \frac{1}{R^d}\int_{\Lambda(R)} E\Big[\mathrm{Tr}\Big[\exp\Big(-\frac{t}{2}\widetilde{H_\omega^{2R,\xi}}\Big)(x,x)\Big]\Big]dx$$

for any t, ξ, $R > 0$, where $\exp(-t\widetilde{H_\omega^{2R,\xi}}/2)(x,y)$, $(t,x,y) \in (0,\infty)\times\Lambda(2R)\times\Lambda(2R)$ *is the integral kernel of the heat semigroup* $\exp(-t\widetilde{H_\omega^{2R,\xi}}/2)$ *generated by* $\widetilde{H_\omega^{2R,\xi}}/2$*. Then there exist finite constants c_1, c_2 and c_3 independent of t, ξ and R such that*

$$\Big|\widetilde{N}\Big(\frac{t}{2};\xi\Big) - \widetilde{N}^R\Big(\frac{t}{2};\xi\Big)\Big| \le \frac{c_1}{t^h}\exp\Big(c_2(\xi t)^\nu - c_3\frac{R^2}{t}\Big) \tag{3.2}$$

for any t, ξ, $R > 0$.

The proof of this lemma is same with that of Lemma 3.1 in [53]. The fundamental tool is the representation of $\exp(-t\widetilde{H_\omega^{2R,\xi}}/2)(x,y)$ by the Feynman-Kac-Itô formula: let $w = (w^1(t), w^2(t), \ldots, w^d(t))$ be a d-dimensional Wiener process starting at 0 and $M_x^\xi(t)$ be the solution of the ordinary differential equation

$$\begin{cases} \dfrac{d}{dt}M_x^\xi(t) = \xi M_x^\xi(t)\Xi_x(t), \\ M_x^\xi(0) = I \end{cases} \tag{3.3}$$

on the space $\mathrm{End}(V)$ of the endomorphisms on V, where

$$\Xi_x(t) := -\frac{1}{2}\sum_{j<k} i\gamma_j\gamma_k B_{\omega,k}^j(x + w(t)).$$

Then $\exp(-t\widetilde{H_\omega^{2R,\xi}}/2)(x,y)$ is represented as

$$\begin{aligned}
&\exp(-t\widetilde{H_\omega^{2R,\xi}}/2)(x,y) \\
&= E^w\Big[\exp\Big(-i\xi\sum_{j=1}^{d}\int_0^t A_\omega^{2R,j}(x+w(s))dw^j(s)\Big)M_x^\xi(t) \\
&\qquad \times\chi\{x+w(s)\in\Lambda(2R) \text{ for any } 0\le s\le t\}\Big|x+w(t)=y\Big] \qquad (3.4)\\
&\qquad \times\frac{1}{(2\pi t)^h}\exp\left(-\frac{|x-y|^2}{2t}\right),
\end{aligned}$$

where E^w is the expectation with respect to a d-dimensional Wiener process, $dw^j(s)$ is the Itô stochastic differential, and, for each Borel set A in the Wiener space, $\chi(A)$ is its indicator function (cf. [12], [33], [52]). As in (3.4) of [53] we have

$$E^\omega[\|M_x^\xi(t)\|_2^p]\le c_1^p\exp(c_2(p\xi t)^\nu) \qquad (3.5)$$

for any $p\ge 1$, where $\|\cdot\|_2$ is the Hilbert-Schmidt norm.

We now proceed to prove the upper estimate. By using

$$\begin{aligned}
\overset{\circ}{A}{}_\omega^{2R,j}(x) &:= \frac{-\Gamma(h)}{2\pi^h}\int\sum_{k\neq j}\frac{x^k-y^k}{|x-y|^d}\{d\rho(y/(2R))\overset{\circ}{A}_\omega(y)\}_k^j dy,\\
\overset{\circ}{A}{}_\omega^{j}(x) &:= \sum_{k=1}^{d}B_{\omega,j}^k(0)\frac{x^k}{2}
\end{aligned}$$

and $\overset{\circ}{M}{}^{\xi}(t) := \exp(\xi t \Xi_0(0))$, we decompose $\widetilde{N}^R(t/2;\xi)$ as the sum of the following three terms:

$$\widetilde{N}_0^R\Big(\frac{t}{2};\xi\Big) := \int_{\Lambda(R)} \frac{dx}{R^d} E\Big[\exp\Big(-i\xi\sum_{j=1}^d \int_0^t \overset{\circ}{A}{}_\omega^{2R,j}(x+w(s))dw^j(s)\Big)\mathrm{Tr}[\overset{\circ}{M}{}^{\xi}(t)]$$
$$\times\chi\{x+w(s)\in\Lambda(2R) \text{ for any } 0\le s\le t\}\Big| w(t)=0\Big]\frac{1}{(2\pi t)^h},$$

$$\widetilde{N}_1^R\Big(\frac{t}{2};\xi\Big) := \int_{\Lambda(R)} \frac{dx}{R^d} E\Big[\exp\Big(-i\xi\sum_{j=1}^d \int_0^t \overset{\circ}{A}{}_\omega^{2R,j}(x+w(s))dw^j(s)\Big)$$
$$\times\mathrm{Tr}[M_x^{\xi}(t)-\overset{\circ}{M}{}^{\xi}(t)]$$
$$\times\chi\{x+w(s)\in\Lambda(2R) \text{ for any } 0\le s\le t\}\Big| w(t)=0\Big]\frac{1}{(2\pi t)^h}$$

and

$$\widetilde{N}_2^R\Big(\frac{t}{2};\xi\Big) := \int_{\Lambda(R)} \frac{dx}{R^d} E\Big[\Big(\exp\Big(-i\xi\sum_{j=1}^d \int_0^t A_\omega^{2R,j}(x+w(s))dw^j(s)\Big)$$
$$-\exp\Big(-i\xi\sum_{j=1}^d \int_0^t \overset{\circ}{A}{}_\omega^{2R,j}(x+w(s))dw^j(s)\Big)\Big)\mathrm{Tr}[M_x^{\xi}(t)]$$
$$\times\chi\{x+w(s)\in\Lambda(2R) \text{ for any } 0\le s\le t\}\Big| w(t)=0\Big]\frac{1}{(2\pi t)^h}.$$

Since the magnetic field for $A^{\circ 2R}_{\omega}(x)$ is constant on $\Lambda(2R)$, we have

$$\widetilde{N}_0^R\Big(\frac{t}{2};\xi\Big) = \int_{\Lambda(R)} \frac{dx}{R^d} E\Big[\exp\Big(i\frac{\xi}{2}\sum_{j,k=1}^d B_{\omega,j}^k(0)\int_0^t w(s)^k dw^j(s)\Big)\mathrm{Tr}[\overset{\circ}{M}{}^{\xi}(t)]$$
$$\times\chi\{x+w(s)\in\Lambda(2R) \text{ for any } 0\le s\le t\}\Big| w(t)=0\Big]\frac{1}{(2\pi t)^h}.$$

By the same argument in the proof of Lemma 3.1, we can estimate this as

$$\Big|\widetilde{N}_0^R\Big(\frac{t}{2};\xi\Big) - \widetilde{N}_0\Big(\frac{t}{2};\xi\Big)\Big| \le \frac{c_1}{t^h}\exp\Big(c_2(\xi t)^\nu - \frac{R^2}{2t}\Big), \tag{3.6}$$

where

$$\widetilde{N}_0\Big(\frac{t}{2};\xi\Big) = E\Big[\exp\Big(i\frac{\xi}{2}\sum_{j,k=1}^d B_{\omega,j}^k(0)\int_0^t w(s)^k dw^j(s)\Big)\mathrm{Tr}[\overset{\circ}{M}{}^{\xi}(t)]\Big| w(t)=0\Big]\frac{1}{(2\pi t)^h}.$$

For each ω, there exists an orthogonal matrix U_ω such that $U_\omega^* B_\omega(0) U_\omega$ is the direct sum of $B_\omega^{(\ell)} J$, $\ell = 1, 2, \ldots, h$, where $B_\omega^{(1)}, \ldots, B_\omega^{(h)} \geq 0$ and J is the matrix defined in (2.8). Since the system of the Hermitian matrices $\{\widehat{\gamma}_k := \sum_j U_{\omega,k}^j \gamma_j\}$ also satisfies the relation (2.2), we have

$$\mathrm{Tr}[\overset{\circ}{M}{}^{\xi}(t)] = \prod_{\ell=1}^{h} 2\cosh\Big(\frac{\xi t}{2} B_\omega^{(\ell)}\Big).$$

Moreover, by the $O(d)$-invariance of the Wiener measure and the formula for the stochastic area due to P. Lévy (cf.[33] VI-(6.10)), we have

$$E^w\Big[\exp\Big(i\frac{\xi}{2}\sum_{j,k=1}^{d} B_{\omega,j}^k(0)\int_0^t w(s)^k dw^j(s)\Big)\Big| w(t) = 0\Big] = \prod_{\ell=1}^{h} \frac{\xi t B_\omega^{(\ell)}/2}{\sinh(\xi t B_\omega^{(\ell)}/2)}.$$

Thus we have

$$\frac{\widetilde{N}_0(t/2;\xi)}{\xi^h} = E\Big[\prod_{\ell=1}^{h} \frac{B_\omega^{(\ell)}}{2\pi}\coth\Big(\frac{\xi t B_\omega^{(\ell)}}{2}\Big)\Big]. \tag{3.7}$$

For $\widetilde{N}_1^R(t/2;\xi)$, we use

$$M_x^\xi(t) - \overset{\circ}{M}{}^{\xi}(t) = \xi\int_0^t M_x^\xi(s)(\Xi_x(s) - \Xi(0))\overset{\circ}{M}{}^{\xi}(t-s)ds$$

and

$$|\mathrm{Tr}[M_x^\xi(t) - \overset{\circ}{M}{}^{\xi}(t)]| \leq \xi\int_0^t ds\|\Xi_x(s) - \Xi_0(0)\|_{op}\|\overset{\circ}{M}{}^{\xi}(t-s)\|_2\|M_x^\xi(s)\|_2,$$

where $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_{op}$ are the trace norm, the Hilbert-Schmidt norm and the operator norm, respectively. Then we have

$$\Big|\widetilde{N}_1^R\Big(\frac{t}{2};\xi\Big)\Big| \leq \frac{c_1\xi}{t^{h-1}} \sup_{x\in\Lambda(2R)} E[\|B_\omega(x) - B_\omega(0)\|_2^3]^{1/3}\exp(c_2(\xi t)^\nu). \tag{3.8}$$

For $\widetilde{N}_2^R(t/2;\xi)$, we use Lemma 2.3 in [23] (Lemma 2.2 in [42]) as follows:

$$E\Big[\Big|\sum_{j=1}^d \int_0^t A_\omega^{2R,j}(x+w(s))dw^j(s) - \sum_{j=1}^d \int_0^t \overset{\circ}{A}{}_\omega^{2R,j}(x+w(s))dw^j(s)\Big|^2$$

$$\times\chi\{x+w(s)\in\Lambda(2R) \text{ for any } 0\le s\le t\}\Big|w(t)=0\Big] \tag{3.9}$$

$$\le E\Big[\Big|\sum_{j=1}^d \int_0^t (A_\omega^{2R,j} - \overset{\circ}{A}{}_\omega^{2R,j})(x+w(s))$$

$$\times\chi_{\Lambda(2R)}(x+w(s))dw^j(s)\Big|^2\Big|w(t)=0\Big]$$

$$\le ct^{3/4}E\Big[\int_0^t |(A_\omega^{2R} - \overset{\circ}{A}{}_\omega^{2R})(x+w(s))|^8\chi_{\Lambda(2R)}(x+w(s))ds\Big|w(t)=0\Big]^{1/4},$$

where $\chi_A(x)$ is the indicator function on $\mathbb{R}^d$ for a Borel set A in $\mathbb{R}^d$. By the definition (3.1) of the vector potential we have

$$E^\omega[|(A_\omega^{2R} - \overset{\circ}{A}{}_\omega^{2R})(x+w(s))|^8] \le cR^8 \sup_{y\in\Lambda(4R)} E[\|B_\omega(y)-B_\omega(0)\|_2^8].$$

Thus we have

$$\Big|\widetilde{N}_2^R\Big(\frac{t}{2};\xi\Big)\Big| \le \frac{c_1\xi R}{t^{h-1/2}} \sup_{y\in\Lambda(4R)} E[\|B_\omega(y)-B_\omega(0)\|^8]^{1/8}\exp(c_2(\xi t)^\nu). \tag{3.10}$$

By (3.2), (3.6), (3.7), (3.8) and (3.10), we obtain

$$\frac{\widetilde{N}(t/2;\xi)}{\xi^h} \le E\Big[\prod_{\ell=1}^h \frac{B_\omega^{(\ell)}}{2\pi}\coth\Big(\frac{\xi t B_\omega^{(\ell)}}{2}\Big)\Big] + \frac{c_1}{(\xi t)^h}\exp\Big(c_2(\xi t)^\nu - \frac{R^2}{2t}\Big)$$

$$+\frac{c_3}{(\xi t)^{h-1}}\Big(1+\frac{R}{\sqrt{t}}\Big)\mathcal{B}(R)\exp(c_4(\xi t)^\nu), \tag{3.11}$$

where $\mathcal{B}(R)$ is a strictly increasing continuous function such that $\mathcal{B}(0)=0$ and

$$\mathcal{B}(R) \ge \sup_{x\in\Lambda(4R)} E[\|B_\omega(x)-B_\omega(0)\|^8]^{1/8}.$$

The function $\mathcal{B}(R)$ exists because of the conditions (b-ii) and (b-v). We will take the limit as $\xi\to\infty$, $t\to 0$ and $R\to 0$ so that $\xi t\to\infty$, $c_2(\xi t)^\nu - R^2/(2t) = 0$ and

$\mathcal{B}(R)\exp(c_4(\xi t)^\nu) = \exp(-c_2(\xi t)^\nu)$, where c_2 and c_4 are the constants in (3.11). It is possible, for example, by setting

$$R = R(\Xi) = \mathcal{B}^{-1}(\exp(-(c_2+c_4)\Xi^\nu)),$$

$$t = t(\Xi) = \mathcal{B}^{-1}(\exp(-(c_2+c_4)\Xi^\nu))^2 \Big/ (2c_2\Xi^\nu),$$

$$\xi = \xi(\Xi) = 2c_2\Xi^{\nu+1} \Big/ \mathcal{B}^{-1}(\exp(-(c_2+c_4)\Xi^\nu))^2$$

and letting $\Xi \to \infty$. Then we obtain

$$\limsup_{\Xi\to\infty} \frac{\widetilde{N}(t(\Xi)/2;\xi(\Xi))}{\xi(\Xi)^h} \le E\Big[\prod_{\ell=1}^h \frac{B_\omega^{(\ell)}}{2\pi}\Big] = \frac{E[\sqrt{\det B_\omega(0)}]}{(2\pi)^h}.$$

Since $\widetilde{N}(t/2;\xi) \ge e^{-t\lambda/2}N(\lambda;\xi)$ for each $\lambda > 0$, we obtain

$$\limsup_{\xi\uparrow\infty} \frac{N(\lambda;\xi)}{\xi^h} \le \frac{E[\sqrt{\det B_\omega(0)}]}{(2\pi)^h}.$$

4. **Proof of Theorems 4**

In this section we assume the conditions (b-i)–(b-iv).

We first prove the following:

Lemma 4.1. *$\widetilde{N}^+(t) - \widetilde{N}^-(t)$ is independent of $t > 0$, where $\widetilde{N}^\pm(t) := \int_0^\infty e^{-t\lambda}dN^\pm(\lambda)$.*

Proof. Since the condition (b-v) is replaced by (b-iv), Lemma 3.1 is modified as follows: there exist $t_0 > 0$ and c_1, c_2 such that

$$\Big|\widetilde{N}^\pm\Big(\frac{t}{2}\Big) - \widetilde{N}^{\pm,R}\Big(\frac{t}{2}\Big)\Big| \le \frac{c_1}{t^h}\exp\Big(-c_2\frac{R^2}{t}\Big) \tag{4.1}$$

for any $0 \le t \le t_0$, where

$$\widetilde{N}^{\pm,R}\Big(\frac{t}{2}\Big) := \frac{1}{R^d}\int_{\Lambda(R)} E\Big[\mathrm{Tr}\Big[\exp\Big(-\frac{t}{2}\widetilde{H_\omega^{2R}}\Big)(x,x)\Gamma_\pm\Big]\Big]dx$$

and $\widetilde{H^{2R}_\omega}$ is the operator $\widetilde{H^{2R,\xi}_\omega}$ with $\xi = 1$. We next define the operator $\widehat{H^{4R}_\omega}$ on $L^2(\mathbb{R}^d \to V)$ by replacing A by A^{4R} in the definition of H_ω, set

$$\widetilde{N}^{\pm,R}_f\Big(\frac{t}{2}\Big) = E\Big[\mathrm{Tr}\Big[f \exp\Big(-\frac{t}{2}\widehat{H^{4R}_\omega}\Big)\Gamma_\pm f\Big]\Big]$$

for any $f \in C(\mathbb{R}^d \to [0,\infty))$ such that $\int f^2 dx = 1$, and show the following:

$$\Big|\widetilde{N}^{\pm,R}\Big(\frac{t}{2}\Big) - \widetilde{N}^{\pm,R}_f\Big(\frac{t}{2}\Big)\Big| \le \frac{c_3}{t^h}\Big\{\exp\Big(-c_4\frac{R^2}{t}\Big) + \int_{\Lambda(R)^c} f^2(x)dx\Big\} \tag{4.2}$$

for any $0 \le t \le t_0$. For this we use the Feynman-Kac-Itô formula and Mercer's expansion theorem to rewrite $\widetilde{N}^{\pm,R}_f(t/2)$ as

$$\int_{\mathbb{R}^d} E\Big[\mathrm{Tr}\Big[\exp\Big(-\frac{t}{2}\widehat{H^{4R}_\omega}\Big)(x,x)\Gamma_\pm\Big]\Big] f(x)^2 dx.$$

By using

$$\widetilde{N}^{\pm,R}\Big(\frac{t}{2}\Big) \le \sup_{y\in\Lambda(R)} E\Big[\mathrm{Tr}\Big[\exp\Big(-\frac{t}{2}\widetilde{H^{2R}_\omega}\Big)(y,y)\Gamma_\pm\Big]\Big]$$

and the nonnegativity of $\mathrm{Tr}[\exp(-t\widetilde{H^{2R}_\omega})(y,y)\Gamma_\pm]$ and $\mathrm{Tr}[\exp(-t\widehat{H^{4R}_\omega})(x,x)\Gamma_\pm]$, we estimate as

$$\begin{aligned}
&\Big|\widetilde{N}^{\pm,R}\Big(\frac{t}{2}\Big) - \widetilde{N}^{\pm,R}_f\Big(\frac{t}{2}\Big)\Big| \\
&\le \int \Big| \sup_{y\in\Lambda(R)} E\Big[\mathrm{Tr}\Big[\exp\Big(-\frac{t}{2}\widetilde{H^{2R}_\omega}\Big)(y,y)\Gamma_\pm\Big]\Big] \\
&\qquad - E\Big[\mathrm{Tr}\Big[\exp\Big(-\frac{t}{2}\widehat{H^{4R}_\omega}\Big)(x,x)\Gamma_\pm\Big]\Big]\Big| f(x)^2 dx \\
&\le \sup_{x,y\in\Lambda(R)} \Big| E\Big[\mathrm{Tr}\Big[\exp\Big(-\frac{t}{2}\widehat{H^{4R}_\omega}\Big)(x,x)\Gamma_\pm\Big]\Big] - E\Big[\mathrm{Tr}\Big[\exp\Big(-\frac{t}{2}\widetilde{H^{2R}_\omega}\Big)(y,y)\Gamma_\pm\Big]\Big]\Big| \\
&\qquad + \Big(\int_{\Lambda(R)^c} f^2 dx\Big)\Big(\sup_{x\in\mathbb{R}^d} E\Big[\mathrm{Tr}\Big[\exp\Big(-\frac{t}{2}\widehat{H^{4R}_\omega}\Big)(x,x)\Gamma_\pm\Big]\Big] \\
&\qquad\qquad + \sup_{y\in\Lambda(R)} E\Big[\mathrm{Tr}\Big[\exp\Big(-\frac{t}{2}\widetilde{H^{2R}_\omega}\Big)(y,y)\Gamma_\pm\Big]\Big]\Big).
\end{aligned}$$

As in (4.1) we have

$$\sup_{y\in\Lambda(R)} E\Big[\mathrm{Tr}\Big[\exp\Big(-\frac{t}{2}\widetilde{H^{2R}_\omega}\Big)(y,y)\Gamma_\pm\Big]\Big] \le \frac{c_5}{t^h}$$

and

$$\sup_{x\in\mathbb{R}^d} E\Big[\mathrm{Tr}\Big[\exp\Big(-\frac{t}{2}\widehat{H^{4R}_\omega}\Big)(x,x)\Gamma_\pm\Big]\Big] \le \frac{c_6}{t^h} \tag{4.3}$$

for any $0\le t\le t_0$. In the representation of the Feynman-Kac-Itô formula, if $x,\, y\in\Lambda(R)$ and $y+w(s)\in\Lambda(2R)$ for any $0\le s\le t$, then $x+w(s)\in\Lambda(4R)$ for any $0\le s\le t$. Thus, by the stationarity of the random magnetic field and the gauge invariance, we have

$$\begin{aligned}
&E\Big[\mathrm{Tr}\Big[\exp\Big(-\frac{t}{2}\widetilde{H^{2R}_\omega}\Big)(y,y)\Gamma_\pm\Big]\Big]\\
&= E\Big[\exp\Big(-i\sum_{j=1}^d\int_0^t A^{4R,j}_\omega(x+w(s))dw^j(s)\Big)\mathrm{Tr}[\widehat{M^{4R}_x}(t)\Gamma_\pm]\\
&\qquad\times\chi\{y+w(s)\in\Lambda(2R)\text{ for some }0\le s\le t\}\Big| w(t)=0\Big]\frac{1}{(2\pi t)^h},
\end{aligned}$$

where $\widehat{M^{4R}_x}(t)$ is the endomorphism valued process obtained by replacing B_ω by B^{4R}_ω in the definition (3.3) of $M^\xi_x(t)$ with $\xi=1$. Therefore we have

$$\begin{aligned}
&E\Big[\mathrm{Tr}\Big[\exp\Big(-\frac{t}{2}\widehat{H^{4R}_\omega}\Big)(x,x)\Gamma_\pm\Big]\Big]-E\Big[\mathrm{Tr}\Big[\exp\Big(-\frac{t}{2}\widetilde{H^{2R}_\omega}\Big)(y,y)\Gamma_\pm\Big]\Big]\\
&= E\Big[\exp\Big(-i\sum_{j=1}^d\int_0^t A^{4R,j}_\omega(x+w(s))dw^j(s)\Big)\mathrm{Tr}[\widehat{M^{4R}_x}(t)\Gamma_\pm]\\
&\qquad\times\chi\{y+w(s)\notin\Lambda(2R)\text{ for some }0\le s\le t\}\Big| w(t)=0\Big]\frac{1}{(2\pi t)^h}.
\end{aligned}$$

Then, by a standard argument on the Wiener process, we have

$$\begin{aligned}
&\Big|E\Big[\mathrm{Tr}\Big[\exp\Big(-\frac{t}{2}\widehat{H^{4R}_\omega}\Big)(x,x)\Gamma_\pm\Big]\Big]-E\Big[\mathrm{Tr}\Big[\exp\Big(-\frac{t}{2}\widetilde{H^{2R}_\omega}\Big)(y,y)\Gamma_\pm\Big]\Big]\Big|\\
&\le\frac{c_7}{t^h}\exp\Big(-c_8\frac{R^2}{t}\Big)
\end{aligned}$$

and we obtain (4.2) for any $0\le t\le t_0$.

We decompose $\widetilde{N^{\pm,R}_f}(t/2)$ as the sum of the following three terms:

$$\begin{aligned}
&\widetilde{N^{\pm,R}_{f,\varepsilon,1}} := E[\mathrm{Tr}[fE([0,\varepsilon]:\widehat{H^{\pm,4R}_\omega})\Gamma_\pm f]],\\
&\widetilde{N^{\pm,R}_{f,\varepsilon,2}}\Big(\frac{t}{2}\Big) := E\Big[\mathrm{Tr}\Big[f\Big\{\exp\Big(-\frac{t}{2}\widehat{H^{\pm,4R}_\omega}\Big)-1\Big\}E([0,\varepsilon]:\widehat{H^{\pm,4R}_\omega})\Gamma_\pm f\Big]\Big]
\end{aligned}$$

and

$$\widetilde{N^{\pm,R}_{f,\varepsilon,3}}(t/2) := E\Big[\mathrm{Tr}\Big[f\exp\Big(-\frac{t}{2}\widehat{H^{\pm,4R}_\omega}\Big)E((\varepsilon,\infty):\widehat{H^{\pm,4R}_\omega})\Gamma_\pm f\Big]\Big]$$

where $\varepsilon\in(0,\infty)$ and $E(\Lambda:\widehat{H^{\pm,4R}_\omega})$, $\Lambda\in\mathcal{B}(\mathbb{R})$, is the resolution of the identity of the Pauli Hamiltonian $\widehat{H^{\pm,4R}_\omega}$. It is easy to show

$$\Big|\widetilde{N^{\pm,R}_{f,\varepsilon,2}}\Big(\frac{t}{2}\Big)\Big| \le (e^{t\varepsilon/2}-1)E\Big[\mathrm{Tr}\Big[f\exp\Big(-\frac{t}{2}\widehat{H^{\pm,4R}_\omega}\Big)f\Big]\Big] \le (e^{t\varepsilon/2}-1)\frac{c_9}{t^h} \tag{4.4}$$

for any $0\le t\le t_0$.

A key estimate is

$$\Big|\widetilde{N^{+,R}_{f,\varepsilon,3}}\Big(\frac{t}{2}\Big)-\widetilde{N^{\pm,R}_{f,\varepsilon,3}}\Big(\frac{t}{2}\Big)\Big| \le \frac{c_{10}}{\varepsilon t^h}\Big(1+\frac{1}{t^2}+\|\nabla f\|^2\Big)^{1/2}\|\nabla f\| \tag{4.5}$$

for any $0\le t\le t_0$, where $\|\cdot\|$ is the L^2-norm. To prove this, we use a theory on the supersymmetry (cf. [18] §6.3): let $\widehat{D^{4R}_\omega}$ be the Dirac operator obtained by replacing A by A^{4R} in the definition of D_ω. Then $(\widehat{H^{4R}_\omega},\Gamma,\widehat{D^{4R}_\omega})$ has supersymmetry in the sense of §6.3 in [18]. As in the proof of Theorem 6.3 in [18], $\widehat{D^{4R}_\omega}$ is invertible on $\mathrm{Ran}E((\varepsilon,\infty):\widehat{H^{4R}_\omega})$ and

$$\begin{aligned}
&\exp\Big(-\frac{t}{2}\widehat{H^{+,4R}_\omega}\Big)E((\varepsilon,\infty):\widehat{H^{+,4R}_\omega})\Gamma_+\\
&=(\widehat{D^{4R}_\omega})^{-1}\exp\Big(-\frac{t}{2}\widehat{H^{-,4R}_\omega}\Big)E((\varepsilon,\infty):\widehat{H^{-,4R}_\omega})\Gamma_-\widehat{D^{4R}_\omega}.
\end{aligned}$$

Since

$$\widehat{D^{4R}_\omega}f^2-f^2\widehat{D^{4R}_\omega}=2f\sum_{j=1}^d\gamma_j(i\partial_j f),$$

we have

$$\begin{aligned}
&\widetilde{N^{+,R}_{f,\varepsilon,3}}\Big(\frac{t}{2}\Big)-\widetilde{N^{-,R}_{f,\varepsilon,3}}\Big(\frac{t}{2}\Big)\\
&=2E\Big[\mathrm{Tr}\Big[f(\widehat{D^{4R}_\omega})^{-1}\exp\Big(-\frac{t}{2}\widehat{H^{\pm,4R}_\omega}\Big)E((\varepsilon,\infty):\widehat{H^{-,4R}_\omega})\sum_{j=1}^d\gamma_j(i\partial_j f)\Big]\Big].
\end{aligned}$$

By $(\widehat{D_\omega^{4R}})^{-1} = \widehat{D_\omega^{4R}}(\widehat{H_\omega^{4R}})^{-1}$, we have

$$\begin{aligned}
&\Big|\widetilde{N^{+,R}_{f,\varepsilon,3}}\Big(\frac{t}{2}\Big) - \widetilde{N^{-,R}_{f,\varepsilon,3}}\Big(\frac{t}{2}\Big)\Big| \\
&\le 2E\Big[\Big|\Big|\Big| f\widehat{D_\omega^{4R}} \exp\Big(-\frac{t}{4}\widehat{H_\omega^{-,4R}}\Big) E((\varepsilon,\infty) : \widehat{H_\omega^{-,4R}})\Big|\Big|\Big|_2^2\Big]^{1/2} \\
&\quad \times E\Big[\Big|\Big|\Big| (\widehat{H_\omega^{-,4R}})^{-1} \exp\Big(-\frac{t}{4}\widehat{H_\omega^{-,4R}}\Big) E((\varepsilon,\infty) : \widehat{H_\omega^{-,4R}}) \sum_{j=1}^d \gamma_j(i\partial_j f)\Big|\Big|\Big|_2^2\Big]^{1/2},
\end{aligned}$$

where $|||\cdot|||_2$ is the Hilbert-Schmidt norm of the operators acting on $L^2(\mathbb{R}^d \to V)$. The first factor is estimated as

$$E\Big[\Big|\Big|\Big| f\widehat{D_\omega^{4R}} \exp\Big(-\frac{t}{4}\widehat{H_\omega^{-,4R}}\Big) E((\varepsilon,\infty) : \widehat{H_\omega^{-,4R}})\Big|\Big|\Big|_2^2\Big] \le c_{11}\Big(1 + \frac{1}{t^2} + \|\nabla f\|^2\Big). \tag{4.6}$$

In fact, for any $\Phi \in \mathrm{Ran}E((\varepsilon,\infty) : \widehat{H_\omega^{-,4R}})$, we have

$$\begin{aligned}
\|f\widehat{D_\omega^{4R}}\Phi\|_V^2 &= \langle f\Phi, f\widehat{H_\omega^{-,4R}}\Phi\rangle_V + 2\Big\langle \sum_{j=1}^d \gamma_j(i\partial_j f)\Phi, f\widehat{D_\omega^{4R}}\Phi\Big\rangle_V \\
&\le \frac{1}{2}\|f\Phi\|_V^2 + \frac{1}{2}\|f\widehat{H_\omega^{-,4R}}\Phi\|_V^2 + \frac{1}{\eta}\Big\|\sum_{j=1}^d \gamma_j(i\partial_j f)\Phi\Big\|_V^2 + \eta\|f\widehat{D_\omega^{4R}}\Phi\|_V^2
\end{aligned}$$

for any $\eta \in (0,\infty)$ and

$$\|f\widehat{D_\omega^{4R}}\Phi\|_V^2 \le c_{12}\Big(\|f\Phi\|_V^2 + \|f\widehat{H_\omega^{-,4R}}\Phi\|_V^2 + \Big\|\sum_{j=1}^d \gamma_j(i\partial_j f)\Phi\Big\|_V^2\Big).$$

Then, by $|||\widehat{H_\omega^{-,4R}}\exp(-t\widehat{H_\omega^{-,4R}}/8)||| \le c_{13}/t$ and (4.3), we have (4.6). Similarly we have

$$E\Big[\Big|\Big|\Big| (\widehat{H_\omega^{-,4R}})^{-1} \exp\Big(-\frac{t}{4}\widehat{H_\omega^{-,4R}}\Big) \sum_{j=1}^d \gamma_j(i\partial_j f)\Big|\Big|\Big|_2^2\Big] \le \frac{c_{14}}{\varepsilon t^h}\|\nabla f\|^2.$$

Thus we obtain (4.5).

We take the function f as

$$f_a(x) := \frac{1}{(2\pi a)^{h/2}} \exp\Big(-\frac{|x|^2}{4a}\Big)$$

for $a > 0$. Then we have

$$\int_{\Lambda(R)^c} f_a^2(x)dx \le c_{15}\exp\Big(-c_{16}\frac{R^2}{a}\Big).$$

To obtain the decay in (4.2) as $R \to \infty$, we take a as $a(R) = R^{2-\alpha}$ for $\alpha > 0$. Then we have $\|\nabla f_{a(R)}\| = \sqrt{d}/(2R^{1-\alpha/2})$. To obtain the decay in (4.4) and (4.5) as $R \to \infty$, we take ε as $\varepsilon(R) = R^{\alpha/2+\beta-1}$ and take α, $\beta > 0$ so that $\alpha/2 + \beta < 1$. Then we obtain

$$\widetilde{N}^{+}\Big(\frac{t}{2}\Big) - \widetilde{N}^{-}\Big(\frac{t}{2}\Big) = \lim_{R\to\infty} (\widetilde{N}^{+,R}_{f_{a(R)},\varepsilon(R),1} - \widetilde{N}^{-,R}_{f_{a(R)},\varepsilon(R),1})$$

for any $0 \le t \le t_0$. The right hand side is independent of t. By the theorem of the identity of analytic functions, $\widetilde{N}^{+}(t) - \widetilde{N}^{-}(t)$ is independent of all $t > 0$. $\square$

Remark 4.1. If we replace the condition (b-ii) by

(b-ii)$'$ the sample path $x \mapsto B_\omega(x)$ is differentiable,

then we can give a simple proof of Lemma 4.1: in this case, the vector potential A_ω by the Poincaré gauge is differentiable and the integral kernel $\exp(-tH_\omega/2)(x, y)$ is represented by the Feynman-Kac-Itô formula. Then we can show $N^{\pm}(\lambda) = E[\mathrm{Tr}[fE([0, \lambda] : H^{\pm}_\omega)f]]$ for any $f \in C(\mathbb{R}^d \to [0, \infty))$ such that $\int f^2 dx = 1$ (cf. [14] Proposition VI.1.3). As in the proof of (4.5), we have

$$\begin{aligned}
&|(N^{+}(\lambda) - N^{-}(\lambda)) - (N^{+}(\varepsilon) - N^{-}(\varepsilon))| \\
&= |E[\mathrm{Tr}[fE((\varepsilon, \lambda] : H^{+}_\omega)f]] - E[\mathrm{Tr}[fE((\varepsilon, \lambda] : H^{-}_\omega)f]]| \\
&= |2E[\mathrm{Tr}[fD_\omega^{-1}E((\varepsilon, \lambda] : H^{-}_\omega)\sum_{j=1}^{d} \gamma_j(i\partial_j f)]]| \\
&\le \frac{c}{\varepsilon}(1 + \lambda^2 + \|\nabla f\|^2)^{1/2}\|\nabla f\|
\end{aligned}$$

for any $0 < \varepsilon < \lambda$. Since f is arbitrary, $N^{+}(\lambda) - N^{-}(\lambda)$ is independent of $\lambda \ge 0$ and $\widetilde{N}^{+}(t) - \widetilde{N}^{-}(t)$ is independent of $t \ge 0$.

Proof of Theorem 4. The rest of the proof is only to show

$$\lim_{t\downarrow 0}\Big\{\widetilde{N^{+}}\Big(\frac{t}{2}\Big) - \widetilde{N^{-}}\Big(\frac{t}{2}\Big)\Big\} = \Big(\frac{-1}{2\pi}\Big)^{h} E[\mathrm{Pff}(B_\omega(0))].$$

For this we use the heat equation method to prove the index theorem (cf. [18] §12). By (4.1), it is sufficient to show

$$\lim_{t\downarrow 0}\left\{\widetilde{N^{+,1}}\left(\frac{t}{2}\right)-\widetilde{N^{-,1}}\left(\frac{t}{2}\right)\right\}=\left(\frac{-1}{2\pi}\right)^{h}E[\mathrm{Pff}(B_\omega(0))]. \tag{4.7}$$

By the scaling property of the Wiener process, we have

$$\begin{aligned}&\widetilde{N^{+,1}}\left(\frac{t}{2}\right)-\widetilde{N^{-,1}}\left(\frac{t}{2}\right)\\ &=\int_{\Lambda(1)}dxE\left[\exp\Big(-i\sqrt{t}\sum_{j=1}^{d}\int_0^1 A_\omega^{2,j}(x+\sqrt{t}w(s))dw^j(s)\Big)\mathrm{Str}[M_x^{(t)}(1)]\right. \quad (4.8)\\ &\qquad\left.\times\chi\{x+\sqrt{t}w(s)\in\Lambda(2)\text{ for any }0\le s\le 1\}\middle|w(1)=0\right]\frac{1}{(2\pi t)^h},\end{aligned}$$

where, for any $M\in\mathrm{End}(V)$, $\mathrm{Str}[M]=\mathrm{Tr}[M\Gamma]$ is the supertrace, $\{M_x^{(t)}(s):s\ge 0\}$ is the endomorphism valued process obtained by replacing $\Xi_x(s)$ and ξ by

$$\Xi_x^{(t)}(s):=-\frac{1}{2}\sum_{j<k}i\gamma_j\gamma_k B_{\omega,k}^{j}(x+\sqrt{t}w(s))$$

and t, respectively, in the ordinary differential equation (3.3). By using the ordinary differential equation, we have

$$M_x^{(t)}(1)=I+tM_1+t^2M_2+\cdots+t^nM_n+t^{n+1}M_{n,R}$$

for any $n\in\mathbb{N}$, where

$$M_p=\int\cdots\int_{0\le s_p\le\cdots\le s_2\le s_1\le 1}\Xi_x^{(t)}(s_p)\cdots\Xi_x^{(t)}(s_2)\Xi_x^{(t)}(s_1)ds_p\cdots ds_2ds_1$$

for $p=1,2,\dots,n$, and

$$M_{n,R}=\int\cdots\int_{0\le s_{n+1}\le\cdots\le s_2\le s_1\le 1}M_x^{(t)}(s_{n+1})\Xi_x^{(t)}(s_{n+1})\cdots\Xi_x^{(t)}(s_2)\Xi_x^{(t)}(s_1)ds_{n+1}\cdots ds_2ds_1.$$

We now use the Berezin formula:

$$\mathrm{Str}\left[\sum_{p=0}^{d}\sum_{1\le j_1<j_2<\cdots<j_p\le d}C_{j_1j_2\cdots j_p}\gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_p}\right]=(-2i)^hC_{12\cdots d} \tag{4.9}$$

(cf. [18] §12.2). Then we have $\mathrm{Str}[M_p] = 0$ for any $p < h$. Therefore we take $n = h$. Since $E^\omega[\|M_x^{(t)}(s)\|_2^2] \le c$ for $0 \le t \le t_0$ and $0 \le s \le 1$ as in (3.5), we can show that the function obtained by replacing $M_x^{(t)}(1)$ by $t^{h+1}M_{h,R}$ in (4.8) tends to zero as $t \to 0$. Moreover, by Lemma 2.3 in [23] (Lemma 2.2 in [42]), we have

$$\sup_{0<t\le 1} E\Big[\Big|\sum_{j=1}^d \int_0^1 A_\omega^{2,j}(x+\sqrt{t}w(s))dw^j(s)\Big|^2 \Big| w(1)=0\Big] < \infty.$$

Therefore the right hand side of (4.7) equals

$$\lim_{t\downarrow 0} \int_{\Lambda(1)} dx E[\mathrm{Str}[M_h]\chi\{x+\sqrt{t}w(s) \in \Lambda(2) \text{ for any } 0 \le s \le 1\}|w(1)=0]\frac{1}{(2\pi)^h}.$$

Since the integrand is uniformly integrable, this equals

$$E\Big[\mathrm{Str}\Big[\Big(-\frac{1}{2}\sum_{j<k} i\gamma_j\gamma_k B_{\omega,k}^j(0)\Big)^h\Big]\Big]\frac{1}{h!(2\pi)^h}.$$

By using again the Berezin formula (4.9), we can show that this coincides with the right hand side of (4.7). □

5. Proof of Theorem 5: (II) Lower estimate

In this section we assume the conditions (b-i)–(b-iii) and (b-v)–(b-vii). By the orthogonal matrix U_ω in the condition (b-vii)$'$, we define a distance on $\mathbb{R}^d$ by

$$d_\omega(x,y) := \max_{1\le \ell\le h}\Big(\sum_{j=2\ell-1}^{2\ell}\{\sum_{k=1}^d U_{\omega,j}^k(x^k-y^k)\}^2\Big)^{1/2}$$

for any $x,\ y \in \mathbb{R}^d$. For each $p \in \mathbb{R}^d$ and $r > 0$, we define a polydisk by $\mathcal{D}_\omega(p,r) := \{x \in \mathbb{R}^d : d_\omega(p,x) < r\}$. We take an osculatory packing of the unit cube $\Lambda(1)$ by the polydisks inductively so that the related quantities are measurable as follows: we first set $r_1^\omega := \sup\{r > 0 : \mathcal{D}_\omega(p,r) \subset \Lambda(1) \text{ for some } p \in \Lambda(1)\}$, which is measurable in ω. Then the set $S^\omega(1,0) := \{p \in \Lambda(1) : \mathcal{D}_\omega(p,r_1^\omega) \subset \Lambda(1)\}$ is not empty. We next set $p_1^{\omega,1} := \sup\{p^1 : p \in S^\omega(1,0)\}$ and $S^\omega(1,1) := \{p \in S^\omega(1,0) : p^1 = p_1^{\omega,1}\}$. Then $p_1^{\omega,1}$ is measurable in ω and $S^\omega(1,1)$ is not empty. Similarly, when the numbers $p_1^{\omega,1}, p_1^{\omega,2}, \ldots, p_1^{\omega,j-1}$ and the nonempty sets $S^\omega(1,0), S^\omega(1,1), \dots, S^\omega(1,j-1)$ are obtained for some $j \le d$, we set $p_1^{\omega,j} := \sup\{p^j : p \in S^\omega(1,j-1)\}$ and $S^\omega(1,j) := \{p \in S^\omega(1,j-1) : p^j = p_1^{\omega,j}\}$. Then $p_1^{\omega,j}$ is measurable in ω and $S^\omega(1,j)$ is not empty. By continuing this procedure, we obtain a $\Lambda(1)$-valued random variable $p_1^\omega = (p_1^{\omega,1}, p_1^{\omega,2}, \dots, p_1^{\omega,d})$. Similarly when $\Lambda(1)$-valued random variables $p_1^\omega, p_2^\omega, \dots, p_{\nu-1}^\omega$ are obtained for some $\nu \ge 2$, we set

$$r_\nu^\omega := \sup\left\{r > 0 : \mathcal{D}_\omega(p,r) \subset \Lambda(1) - \bigcup_{\mu=1}^{\nu-1} \overline{\mathcal{D}_\mu^\omega} \text{ for some } p \in \Lambda(1)\right\}$$

and

$$S^\omega(\nu,0) := \left\{p \in \Lambda(1) : \mathcal{D}_\omega(p, r_\nu^\omega) \subset \Lambda(1) - \bigcup_{\mu=1}^{\nu-1} \overline{\mathcal{D}_\mu^\omega}\right\},$$

where $\overline{\mathcal{D}_\mu^\omega}$ is the closure of the domain $\mathcal{D}_\mu^\omega := \mathcal{D}_\omega(p_\mu^\omega, r_\mu^\omega)$. Moreover, when the positive numbers $p_\nu^{\omega,1}, p_\nu^{\omega,2}, \dots, p_\nu^{\omega,j-1}$ and the nonempty sets $S^\omega(\nu,0)$, $S^\omega(\nu,1), \dots, S^\omega(\nu,j-1)$ are obtained for some $j \le d$, we set $p_\nu^{\omega,j} := \sup\{p^j : p \in S^\omega(\nu,j-1)\}$ and $S^\omega(\nu,j) := \{p \in S^\omega(\nu,j-1) : p^j = p_\nu^{\omega,j}\}$. By continuing this procedure, we obtain the decreasing sequence of positive random variables $\{r_\nu^\omega\}_{\nu=1}^\infty$ and $\Lambda(1)$-valued random variables $\{p_\nu\}_{\nu=1}^\infty$ so that the sets $\{\mathcal{D}_\nu^\omega\}_{\nu=1}^\infty$ are mutually disjoint and is included in $\Lambda(1)$. As in the usual proof of Vitali's covering theorem, we can show that this packing is a complete packing:

$$\sum_{\nu=1}^\infty |\mathcal{D}_\nu^\omega| = 1 = |\Lambda(1)|, \tag{5.1}$$

where $|\cdot|$ is the volume.

In the following we use the ω-dependent complex coordinate $(z_\omega^1, \dots, z_\omega^h)$ defined by $z_\omega^\ell = x_\omega^{2\ell-1} + i x_\omega^{2\ell}$ and $x_\omega^j = \sum_{k=1}^d U_{\omega,j}^k x^k$. In terms of this coordinate, the polydisk $\mathcal{D}_\omega(p,r)$ is the usual polydisk $\{z_\omega \in \mathbb{C}^h : |z_\omega^\ell - z_\omega^\ell(p)| < r \text{ for } \ell = 1,2,\dots,h\}$ and the cube $\Lambda(1)$ is a domain obtained by rotating $(-1/2, 1/2)^d$. On the other hand. the magnetic field is represented as the complex 2-form

$$\mathbb{B}_\omega(z_\omega) = \frac{1}{2} \sum_{\ell,m} \mathbb{B}_{\omega,m}^\ell(z_\omega) dz_\omega^\ell \wedge d\overline{z_\omega^m}$$

and the vector potential defined in (2.1) is represented as the complex 1-form

$$\mathbb{A}_\omega(z_\omega) = \sum_{\ell=1}^{h} (\mathbb{A}_\omega^\ell(z_\omega) d\overline{z_\omega^\ell} + \overline{\mathbb{A}_\omega^\ell(z_\omega)} dz_\omega^\ell)$$

on $\Lambda(R_0)$, where R_0 and $(\mathbb{B}_{\omega,m}^\ell(z_\omega))_{1\le\ell,m\le h}$ are the positive number and the skew Hermitian matrix valued random field, respectively, defined in Section 2, and

$$\mathbb{A}_\omega^\ell(z_\omega) = \frac{1}{2}\int_0^1 \sum_{m=1}^{h} \mathbb{B}_{\omega,\ell}^m(tz_\omega) t z_\omega^m dt.$$

The Hermitian matrices $\gamma_{\omega,1}, \gamma_{\omega,2}, \dots, \gamma_{\omega,d}$ defined by $\gamma_{\omega,j} = \sum_{k=1}^d U_{\omega,j}^k \gamma_k$ also satisfies the commutation relation (2.2). We represent these matrices as $\gamma_{\omega,2\ell-1} = \mathrm{ext}(\delta_\ell) + \mathrm{int}(\delta_\ell)$, $\gamma_{\omega,2\ell} = i(\mathrm{ext}(\delta_\ell) - \mathrm{int}(\delta_\ell))$ for $\ell = 1, 2, \dots, h$, on $V = \Lambda(\mathbb{C}^h)$, where $\{\delta_1, \delta_2, \dots, \delta_h\}$ is a unitary basis of $\mathbb{C}^h$, ext is the exterior multiplication, i.e., $\mathrm{ext}(\delta_\ell)\psi = \delta_\ell \wedge \psi$ and int is the interior multiplication, i.e., the adjoint of ext. Then the Dirac operator is represented as

$$D_\omega^\xi = \sum_{\ell=1}^{h} \left\{ \mathrm{ext}(\delta_\ell) 2\Big(i\frac{\partial}{\partial \overline{z_\omega^\ell}} + \xi \mathbb{A}_\omega^\ell(z_\omega)\Big) + \mathrm{int}(\delta_\ell) 2\Big(i\frac{\partial}{\partial z_\omega^\ell} + \xi \overline{\mathbb{A}_\omega^\ell(z_\omega)}\Big) \right\} \tag{5.2}$$

on $\Lambda(R_0)$.

Until (5.12) below we fix ω and omit to indicate the ω-dependence. For each $\nu = 1, 2, \dots$ and $0 < R < R_0$, we introduce a vector potential

$$\mathbb{A}^{\nu,R}(z) = \sum_{\ell=1}^{h} (\mathbb{A}^{\nu,R,\ell}(z) d\overline{z^\ell} + \overline{\mathbb{A}^{\nu,R,\ell}(z)} dz^\ell)$$

by $\mathbb{A}^{\nu,R,\ell}(z) := \mathring{\mathbb{A}}^{\nu,R,\ell}(z) + \mathbb{A}^{\Delta,\ell}(z)$, where

$$\mathring{\mathbb{A}}^{\nu,R,\ell}(z) := iB^{(\ell)}(z^\ell - Rp_\nu^\ell)/4,$$

$$\mathbb{A}^{\Delta,\ell}(z) := \frac{1}{2}\int_0^1 \sum_{m=1}^{h} \mathbb{B}_\ell^{\Delta,m}(tz) t z^m dt$$

and $\mathbb{B}^\Delta(z) := \mathbb{B}(z) - \mathbb{B}(0)$. Let $\widehat{D}^{\nu,R,\xi}$ be the Dirac operator obtained by replacing $\mathbb{A}^\ell$ by $\mathbb{A}^{\nu,R,\ell}$ in the definition of the operator D^ξ. Let $H^{\nu,R,\xi}$ be the self-adjoint operator on the space $L^2(R\mathcal{D}_\nu \to V)$ corresponding to the closure of the quadratic form $q^\xi(\Phi, \Psi)$ with the domain $C_0^\infty(R\mathcal{D}_\nu \to V)$ and $\widehat{H}^{\nu,R,\xi}$ be the operator obtained by replacing D^ξ by $\widehat{D}^{\nu,R,\xi}$ in the definition of the operator $H^{\nu,R,\xi}$. Then $H^{\nu,R,\xi}$ and $\widehat{H}^{\nu,R,\xi}$ are unitarily equivalent by the gauge invariance.

We use a uniform estimate of the solution of the $\overline{\partial}$-equation by Berndtsson [9]:

Lemma 5.1. *Let $B(r) := \{z \in \mathbb{C}^h : |z| < r\}$ for $r > 0$. Then, for any $R > 0$ such that $2R\sqrt{d} < R_0$, there exists a function ψ_R on $B(R\sqrt{d})$ such that*

$$\overline{\partial}\psi_R = i\sum_{\ell=1}^{h} \mathbb{A}^{\Delta,\ell}(z)d\overline{z^\ell} \tag{5.3}$$

and

$$\sup_{z\in\Lambda(R)} |\psi_R(z)| \le C_d R^2 \sup_{z\in B(R\sqrt{d})} \max_{1\le\ell,m\le h} |\mathbb{B}_m^{\Delta,\ell}(z)|,$$

where C_d depends only on the dimension.

Proof. By Theorem 2 in Berndtsson [9], there exists a function Ψ_R on $B(1)$ such that

$$\overline{\partial}\Psi_R = i\sum_{\ell=1}^{h} \mathbb{A}^{\Delta,\ell}(R\sqrt{d}z)d\overline{z^\ell}$$

on $B(1)$ and

$$\sup_{z\in B(1/2)} |\Psi_R(z)| \le C_d' \sup_{x\in B(1)} \max_{1\le\ell\le h} |\mathbb{A}^{\Delta,\ell}(R\sqrt{d}z)|.$$

We set $\psi_R(z) := R\sqrt{d}\Psi_R(z/(R\sqrt{d}))$. Then (5.3) holds and

$$\begin{aligned}\sup_{z\in B(R\sqrt{d}/2)} |\psi_R(z)| &\le C_d' R\sqrt{d} \sup_{z\in B(R\sqrt{d})} \max_{1\le\ell\le h} |\mathbb{A}^{\Delta,\ell}(z)| \\ &\le C_d R^2 \sup_{z\in B(R\sqrt{d})} \max_{1\le\ell,m\le h} |\mathbb{B}_m^{\Delta,\ell}(z)|.\end{aligned}$$

□

We now take $R > 0$ so that $2R\sqrt{d} < R_0$, and set

$$\psi_{\nu,R}(z) := -\sum_{\ell=1}^{h} \frac{B^{(\ell)}}{4}|z^\ell - Rp_\nu^\ell|^2 + \psi_R(z)$$

and $\phi_{\nu,R}(z) := \exp(\xi\psi_{\nu,R}(z))$. Then we have

$$\widehat{D}^{\nu,R,\xi}\Big(\prod_{\ell=1}^{h}(z^\ell - Rp_\nu^\ell)^{n(\ell)}\Big)\phi_{\nu,R}(z) = 0$$

for any $\mathbf{n} = (n(1), n(2), \cdots, n(h)) \in \mathbb{Z}_+^h$. To obtain functions in the domain of $\widehat{H}^{\nu,R,\xi}$, we introduce $\zeta \in C_0^\infty(\mathbb{C} \to [0,1])$ such that

$$\zeta(z) = \begin{cases} 0 \text{ if } z \notin B^2(1), \\ 1 \ \text{ if } z \in B^2(1-\delta) \end{cases} \tag{5.4}$$

and $|\nabla\zeta| \le c/\delta$, where $B^2(r) := \{x \in \mathbb{C} : |z| < r\}$ for $r > 0$, $0 < \delta < 1/2$ is an arbitrary small constant and c is a constant independent of δ. For any $\mathbf{n} = (n(1), n(2), \cdots, n(h)) \in \mathbb{Z}_+^h$, we set

$$\phi_{\nu,\mathbf{n}}(z) := \Big(\prod_{\ell=1}^{h} \zeta\Big(\frac{z^\ell - Rp_\nu^\ell}{Rr_\nu}\Big)(z^\ell - Rp_\nu^\ell)^{n(\ell)}\Big)\phi_{\nu,R}(z).$$

Then we have

$$\begin{aligned} \|\widehat{D}^{\nu,R,\xi}\phi_{\nu,\mathbf{n}}\|^2 \le \frac{c}{(Rr_\nu\delta)^2} \sum_{\ell=1}^{h} \int_{B^2(Rr_\nu)-B^2(Rr_\nu(1-\delta))} |z^1|^{2n(\ell)} \exp(-\xi B^{(\ell)}|z^1|^2/2)dx^1dx^2 \\ \times\Big(\prod_{m\neq\ell}\int \zeta^2\Big(\frac{z^1}{Rr_\nu}\Big)|z^1|^{2n(m)}\exp(-\xi B^{(m)}|z^1|^2/2)dx^1dx^2\Big) \\ \times \exp\Big(2\xi \sup_{z\in\Lambda(R)} |\psi_R(z)|\Big) \end{aligned}$$

and

$$\begin{aligned} \frac{\|\widehat{D}^{\nu,R,\xi}\phi_{\nu,\mathbf{n}}\|^2}{\|\phi_{\nu,\mathbf{n}}\|^2} \le \frac{c}{(Rr_\nu\delta)^2}\exp\Big(4\xi \sup_{z\in\Lambda(R)}|\psi_R(z)|\Big) \\ \times \sum_{\ell=1}^{h}\int_{B^2(Rr_\nu)-B^2(Rr_\nu(1-\delta))} |z^1|^{2n(\ell)}\exp(-\xi B^{(\ell)}|z^1|^2/2)dx^1dx^2 \\ \times\Big\{\int_{B^2(Rr_\nu(1-\delta))}|z^1|^{2n(\ell)}\exp(-\xi B^{(\ell)}|z^1|^2/2)dx^1dx^2\Big\}^{-1}. \end{aligned}$$

We now assume

$$\sup_{z\in B(R\sqrt{d})}\max_{1\le\ell,m\le h}|B_m^{\Delta,\ell}(z)| < \eta.$$

Then we have

$$\frac{\|\widehat{D}^{\nu,R,\xi}\phi_{\nu,\mathbf{n}}\|^2}{\|\phi_{\nu,\mathbf{n}}\|^2} \le \frac{c_1}{(Rr_\nu)^2\delta}\sum_{\ell=1}^{h}\Big\{(1-\delta)^{2n(\ell)}I(n(\ell),\xi B^{(\ell)}(Rr_\nu(1-\delta))^2/2)\exp(-\xi c_2\eta R^2)\Big\}^{-1},$$

where

$$I(n,a) := \int_0^1 s^n \exp(a(1-s))ds \tag{5.5}$$

for $n,\, a \geq 0$.

We fix $\lambda > 0$ arbitrarily. For

$$\|\widehat{D}^{\nu,R,\xi}\phi_{\nu,\mathbf{n}}\|^2/\|\phi_{\nu,\mathbf{n}}\|^2 \leq \lambda \tag{5.6}$$

to hold, it is sufficient that

$$\frac{c_1 h}{(Rr_\nu)^2\delta\lambda(1-\delta)^{2n(\ell)}} \leq I(n(\ell), \xi B^{(\ell)}(Rr_\nu(1-\delta))^2/2)\exp(-\xi c_2\eta R^2), \tag{5.7}$$

for all $\ell \in \{1, 2, \ldots, h\}$. Introducing $0 < \sigma < 1/2$, we use the estimate

$$I(n,a) \geq \begin{cases} \sigma(n/(ae))^n \exp(a(1-\sigma)) & \text{if } \sigma + n/a \leq 1, \\ \sigma(1-\sigma)^n & \text{if } \sigma + n/a \geq 1. \end{cases} \tag{5.8}$$

Then, for (5.7) to hold, it is sufficient that

$$\frac{c_1 h}{(Rr_\nu)^2\delta\lambda\sigma}\Big(\frac{\xi B^{(\ell)}(Rr_\nu)^2 e}{2n(\ell)}\Big)^{n(\ell)} \leq \exp(\xi B^{(\ell)}(Rr_\nu(1-\delta))^2(1-\sigma)/2 - \xi c_2\eta R^2) \tag{5.9}$$

when $n(\ell) \leq \xi B^{(\ell)}(Rr_\nu(1-\delta))^2(1-\sigma)/2$, and

$$\frac{c_1 h}{(Rr_\nu)^2\delta\lambda\sigma((1-\delta)^2(1-\sigma))^{n(\ell)}} \leq \exp(-\xi c_2\eta R^2) \tag{5.10}$$

when $n(\ell) \geq \xi B^{(\ell)}(Rr_\nu(1-\delta))^2(1-\sigma)/2$. However (5.10) is impossible for sufficiently large ξ. Therefore we consider only (5.9). This is rewritten as $g^\nu_\ell(n(\ell)) \leq 0$ and $n(\ell) \leq \xi B^{(\ell)}(Rr_\nu(1-\delta))^2(1-\sigma)/2$, where $g^\nu_\ell(n) := -n\log n + n\mathcal{A}^\nu_\ell - \mathcal{B}^\nu_\ell$ for $n > 0$, $\mathcal{A}^\nu_\ell := \log(\xi B^{(\ell)}(Rr_\nu)^2 e/2)$ and $\mathcal{B}^\nu_\ell := \xi B^{(\ell)}(Rr_\nu(1-\delta))^2(1-\sigma)/2 - \xi c_2\eta R^2 + \log((Rr_\nu)^2\delta\lambda\sigma/(c_1 h))$. The function $g^\nu_\ell(n)$ is increasing on $(0, \exp(\mathcal{A}^\nu_\ell - 1)]$ and its maximum is

$$g^\nu_\ell(\exp(\mathcal{A}^\nu_\ell - 1)) = \exp(\mathcal{A}^\nu_\ell - 1) - \mathcal{B}^\nu_\ell$$

$$= \xi B^{(\ell)}(0)(Rr_\nu)^2(1-(1-\delta)^2(1-\sigma))/2 + \xi c_2\eta R^2 - \log((Rr_\nu)^2\delta\lambda\sigma/(c_1 h)).$$

We now assume $\eta < (r_\nu)^2 B^{(\ell)}(1-(1-\delta)^2(1-\sigma))/(2c_2)$. Since δ, $\sigma < 1/2$, by taking ξ large enough, we may regard that $\lim_{n\downarrow 0} g_\ell^\nu(n) < 0$ and $g_\ell^\nu(\exp(\mathcal{A}_\ell^\nu - 1)) > 0$. We put

$$k_\ell^{\nu,\xi} := \inf\{k > 1 : g_\ell^\nu(\exp(\mathcal{A}_\ell^\nu - 1)/k) \le 0\} \tag{5.11}$$

and take another small $\vartheta > 0$. Then $g_\ell^\nu(n) < 0$ for $n \le \exp(\mathcal{A}_\ell^\nu - 1)/(k_\ell^{\nu,\xi} + \vartheta)$. For $n \ge \exp(\mathcal{A}_\ell^\nu - 1)/(k_\ell^{\nu,\xi} + \vartheta)$, we use $\log n \ge \mathcal{A}_\ell^\nu - 1 - \log(k_\ell^{\nu,\xi} + \vartheta)$ to estimate as $g_\ell^\nu(n) \le n(1 + \log(k_\ell^{\nu,\xi} + \vartheta)) - \mathcal{B}_\ell^\nu$. Therefore, for $g_\ell^\nu(n(\ell)) \le 0$ to hold, it is sufficient that

$$n(\ell) \le \mathcal{B}_\ell^\nu / (1 + \log(k_\ell^{\nu,\xi} + \vartheta)). \tag{5.12}$$

The right hand side is less than $\xi B^{(\ell)}(0)(R r_\nu(1-\delta))^2(1-\sigma)/2$. Consequently, for (5.6) to hold, it is sufficient that (5.12) holds for all $\ell \in \{1, 2, \ldots, h\}$.

Thus the counting function $N_\omega(\lambda; \xi, R\mathcal{D}_\nu^\omega)$ of the eigenvalues of the operator $H_\omega^{\nu,R,\xi}$ is greater than or equal to

$$\prod_{\ell=1}^h \Big(\frac{\mathcal{B}_{\ell,\omega}^\nu}{1 + \log(k_{\ell,\omega}^{\nu,\xi} + \vartheta)} - 1 \Big).$$

On the other hand, $k_{\ell,\omega}^{\nu,\xi}$ in (5.11) is rewritten as

$$k_{\ell,\omega}^{\nu,\xi} = K\Big((1-\delta)^2(1-\sigma) - \frac{2c_2\eta}{B_\omega^{(\ell)}(r_\nu^\omega)^2} + \frac{2}{\xi B_\omega^{(\ell)}(R r_\nu^\omega)^2} \log \frac{(R r_\nu^\omega)^2 \delta\lambda\sigma}{c_1 h} \Big), \tag{5.13}$$

where K is the inverse of the function $F(k) = (1 + \log k)/k$, $k > 1$. Since F is strictly decreasing continuous function and $\lim_{k\downarrow 1} F(k) = 1$, K is also strictly decreasing continuous function on the interval $(0,1)$ and $\lim_{f\uparrow 1} K(f) = 1$. Therefore we obtain

$$\liminf_{\xi\uparrow\infty} \frac{N_\omega(\lambda; \xi, R\mathcal{D}_\nu^\omega)}{\xi^h} \ge \prod_{\ell=1}^h \frac{B_\omega^{(\ell)}(R r_\nu^\omega(1-\delta))^2(1-\sigma)/2 - c_2\eta R^2}{1 + \log K((1-\delta)^2(1-\sigma) - 2c_2\eta/(B_\omega^{(\ell)}(r_\nu^\omega)^2))}$$

for almost all ω. We take another small $\upsilon > 0$ and set $\mu_\omega(\upsilon) := \max\{\mu : r_\mu^\omega \ge \upsilon\}$. By the min-max principle, we have

$$N_\omega^R(\lambda; \xi) \ge \sum_{\nu=1}^{\mu_\omega(\upsilon)} N_\omega(\lambda; \xi, R\mathcal{D}_\nu^\omega).$$

Thus we have

$$\liminf_{\xi\uparrow\infty}\frac{N_\omega^R(\lambda;\xi)}{\xi^h R^d}\geq\sum_{\nu=1}^{\mu_\omega(\upsilon)}|\mathcal{D}_\nu^\omega|\prod_{\ell=1}^h\frac{B_\omega^{(\ell)}(1-\delta)^2(1-\sigma)/(2\pi)-c_2\eta/(\pi\upsilon^2)}{1+\log K((1-\delta)^2(1-\sigma)-c_2\eta/(B_\omega^{(\ell)}\upsilon^2))}.$$

We now take the expectation in ω. By the Akcoglu-Krengel superadditive ergodic theorem (cf. [3], [14], [37]) and Fatou's lemma, we have

$$\liminf_{\xi\uparrow\infty}\frac{N(\lambda;\xi)}{\xi^h}\geq E\Bigg[\sum_{\nu=1}^{\mu_\omega(\upsilon)}|\mathcal{D}_\nu^\omega|\prod_{\ell=1}^h\frac{B_\omega^{(\ell)}(1-\delta)^2(1-\sigma)/(2\pi)-c_2\eta/(\pi\upsilon^2)}{1+\log K((1-\delta)^2(1-\sigma)-c_2\eta/(B_\omega^{(\ell)}\upsilon^2))}$$
$$: B_\omega^{(1)},\dots,B_\omega^{(h)}>\varepsilon,\ \sup_{z\in B(R\sqrt{d})}\max_{1\leq\ell,m\leq h}|B_{\omega,m}^{\Delta,\ell}(z)|<\eta\Bigg],$$

where $\varepsilon>0$ is arbitrary and $0<\eta<\varepsilon\upsilon^2(1-(1-\delta)^2(1-\sigma))/(2c_2)$. By taking the limit as R, η, σ, δ, ε, $\upsilon\to 0$, we obtain

$$\liminf_{\xi\uparrow\infty}\frac{N(\lambda;\xi)}{\xi^h}\geq\frac{E[\sqrt{\det B_\omega(0)}]}{(2\pi)^h}.$$

6. **Proof of Theorem 3**

We first consider the case that the conditions (a-i), (a-ii) and (a-vi) are satisfied. Setting

$$C_\omega^j(x^j)=\int_0^{x^j}dt\int_0^t ds B_\omega^j(s),$$

we take the following vector potential:

$$A_\omega^j(x)=(-1)^j\frac{d}{dx^{o(j)}}C_\omega^{o(j)}(x^{o(j)})=(-1)^j\int_0^{x^{o(j)}}dtB_\omega^{o(j)}(t)$$

for $j=1$, 2, where $o(1)=2$ and $o(2)=1$. Then the functions $\phi_n^\omega(x)=z^n\exp(-C_\omega^1(x^1)-C_\omega^2(x^2))$, $n=0,1,2,\dots$, satisfy $\mathcal{A}_\omega\phi_n^\omega=0$, where $z=x^1+ix^2$ and $\mathcal{A}_\omega$ is the operator defined in (2.13). By a family of functions $\{\zeta_R\}_{R>1}\subset C_0^\infty(\mathbb{R}\to[0,1])$ satisfying

$$\zeta_R(t)=\begin{cases}0\text{ if }|t|\geq R+1,\\ 1\text{ if }|t|\leq R,\end{cases}\tag{6.1}$$

and $\sup_{R,t}|\zeta_R'(t)|<\infty$, we restrict ϕ_n^ω as $\phi_{n,R}^\omega(x)=\zeta_R(x^1)\zeta_R(x^2)\phi_n^\omega(x)$. Then we have

$$\|\mathcal{A}_\omega\phi_{n,R}^\omega\|^2 \le c_1\sum_{j=1}^2\int_{R\le|x^j|\le R+1}dx^j\exp(-2C_\omega^j(x^j))$$
$$\times\int dx^{o(j)}\zeta_R(x^{o(j)})^2\exp(-2C_\omega^{o(j)}(x^{o(j)}))|x|^{2n}.$$

When $R\le|x^j|\le R+1$ and $|x^{o(j)}|\le R+1$, we have $|x|^2\le 5(x^j)^2$. Using also $|x|^2\ge(x^j)^2$ for $\|\phi_{n,R}^\omega\|^2$, we have

$$\frac{\|\mathcal{A}_\omega\phi_{n,R}^\omega\|^2}{\|\phi_{n,R}^\omega\|^2}\le c_1 5^n\sum_{j=1}^2\int_{R\le|t|\le R+1}t^{2n}\exp(-2C_\omega^j(t))dt\Big/\int_{|t|\le R}t^{2n}\exp(-2C_\omega^j(t))dt.$$

By the uniform estimate, we have

$$\int_{R\le|t|\le R+1}t^{2n}\exp(-2C_\omega^j(t))dt\le 2(R+1)^{2n}\exp(-2\inf_{R\le|t|\le R+1}C_\omega^j(t)).$$

Moreover, by restricting the integral to $\eta R/2\le|t|\le\eta R$, we have

$$\int_{|t|\le R}t^{2n}\exp(-2C_\omega^j(t))dt\ge\frac{(\eta R)^{2n+1}}{2n+1}\exp\Big(-2\sup_{\eta R/2\le|t|\le\eta R}C_\omega^j(t)\Big),$$

where $0<\eta<1$ is specified later. To estimate these quantities, we use the process $D_\omega^j(t):=C_\omega^j(t)/t^2$ instead of $C_\omega^j(t)$, since the covariance of $D_\omega^j(t)$ is bounded in t.

We now estimate the probability

$$p_j(a_1,a_2):=P\Big(\inf_{R\le|t|\le R+1}D_\omega^j(t)\ge a_1\text{ and }\sup_{\eta R/2\le|t|\le\eta R}D_\omega^j(t)\le a_2\Big)$$

from below for any a_1, $a_2\ge0$. Let $D_\omega^{j,k}(t)=(D_\omega^j(t)+(-1)^kD_\omega^j(-t))/2$, $k=1,2$, be the even and odd parts of $D_\omega^j(t)$, respectively. Then we have

$$p_j(a_1,a_2)\ge P\Big(\inf_{R\le t\le R+1}D_\omega^{j,1}(t)\ge a_1+a_3\text{ and }\sup_{\eta R/2\le t\le\eta R}D_\omega^{j,1}(t)\le a_2-a_3\Big)$$
$$\times P\Big(\sup_{\eta R/2\le t\le\eta R\text{ or }R\le t\le R+1}|D_\omega^{j,2}(t)|\le a_3\Big)$$

for any $a_3\ge0$, since $D_\omega^{j,1}$ and $D_\omega^{j,2}$ are independent as random processes. We decompose the process $D_\omega^{j,1}$ as the sum of $\overset{\circ}{D}{}_\omega^{j,1}(t)=E[D_\omega^{j,1}(t)|D_\omega^{j,1}(R)]$ and $\overline{D_\omega^{j,1}}(t)=D_\omega^{j,1}(t)-\overset{\circ}{D}{}_\omega^{j,1}(t)$.

These processes are independent and $\overset{\circ}{D}{}^{j,1}_{\omega}(t) = X^j(t)W^j_\omega$, where W^j_ω is a random variable obeying the standard normal distribution and $X^j(t) = E[D^{j,1}_\omega(t)D^{j,1}_\omega(R)]/E[D^{j,1}_\omega(R)^2]^{1/2}$.

In the following we assume $0 < r < 1/3$. This is always possible under the condition (a-vi). By the representation

$$E[D^{j,1}_\omega(t)D^{j,1}_\omega(s)] \tag{6.2}$$
$$= \frac{1}{2}\int_0^1 du_1 \int_0^{u_1} du_2 \int_0^1 dv_1 \int_0^{v_1} dv_2 \{\beta^j(tu_2 - sv_2) + \beta^j(tu_2 + sv_2)\}$$

and the condition $\inf_{|x^j|\le r} \beta^j > 0$, we have

$$\begin{aligned} E[D^{j,1}_\omega(t)D^{j,1}_\omega(R)] &\ge c_2 \int_0^1 du_1 \int_0^{u_1} du_2 \int_0^1 dv_1 \int_0^{v_1} dv_2 \chi(|tu_2 - Rv_2| \le r) \\ &= c_2 \int_0^1 du(1-u) \int_0^1 dv(1-v)\chi(|tu - Rv| \le r) \end{aligned}$$

for $R \le t \le R+1$. Then, by a simple calculation, we have

$$\inf_{R>1} \inf_{R\le t\le R+1} RE[D^{j,1}_\omega(t)D^{j,1}_\omega(R)] > 0.$$

On the other hand, by (6.2), we have

$$\begin{aligned} RE[D^{j,1}_\omega(R)^2] &= \frac{1}{2}\int_0^1 du_1 \int_0^{u_1} du_2 \int_0^1 dv_1 \int_0^{Rv_1} dv_2 \{\beta^j(Ru_2 - v_2) + \beta^j(Ru_2 + v_2)\} \\ &\to \frac{1}{2}\int_0^1 du_1 \int_0^{u_1} du_2 \int_{u_2}^1 dv_1 \int_{\mathbb{R}} \beta^j(t)dt = \frac{1}{6}\int_{\mathbb{R}} \beta^j(t)dt \end{aligned}$$

as $R \to \infty$. Therefore we have

$$\sup_{R>1} RE[D^{j,1}_\omega(R)^2] < \infty$$

and

$$\inf_{R\le t\le R+1} X^j(t) \ge \frac{c_3}{\sqrt{R}}$$

for any $R > 1$. Using also the Schwarz inequality, we have

$$\sup_{\eta R/2\le t\le R+1} X^j(t) \le \sup_{\eta R/2\le t\le R+1} E[D^{j,1}_\omega(t)^2]^{1/2} \le \frac{c_4}{\sqrt{\eta R}}$$

for any $R > 1$. Therefore we have

$$p_j(a_1, a_2) \geq P\Big(\frac{\sqrt{R}}{c_3}(a_1 + a_3 + a_4) \leq W_\omega^j \leq \frac{\sqrt{\eta R}}{c_4}(a_2 - a_3 - a_4)\Big)$$
$$\times P\Big(\sup_{\eta R/2 \leq t \leq \eta R \text{ or } R \leq t \leq R+1} |\overline{D_\omega^{j,1}}(t)| \leq a_4\Big)$$
$$\times P\Big(\sup_{\eta R/2 \leq t \leq \eta R \text{ or } R \leq t \leq R+1} |D_\omega^{j,2}(t)| \leq a_3\Big)$$

for any $a_4 \geq 0$.

By Fernique's theorem [26], we have

$$P\Bigg(\sup_{\eta R/2 \leq t \leq \eta R} |D_\omega^{j,2}(t)| \leq \Xi\Big\{\sup_{\eta R/2 \leq t \leq \eta R} E[D_\omega^{j,2}(t)^2]^{1/2} + 2\int_1^\infty \varphi(2^{-u^2})du\Big\}\Bigg)$$
$$\leq 40\int_\Xi^\infty e^{-u^2/2}du$$

for any $\Xi \geq \sqrt{1 + 8\log 2}$, where

$$\varphi(h) = \sup\{E[(D_\omega^{j,2}(t) - D_\omega^{j,2}(s))^2]^{1/2} : \eta R/2 \leq t, s \leq \eta R, |t - s| \leq \eta Rh/2\}.$$

By the representation

$$E[D_\omega^{j,2}(t)D_\omega^{j,2}(s)]$$
$$= \frac{1}{2}\int_0^1 du_1 \int_0^{u_1} du_2 \int_0^1 dv_1 \int_0^{v_1} dv_2 \{\beta^j(tu_2 - sv_2) - \beta^j(tu_2 + sv_2)\},$$

we have

$$E[D_\omega^{j,2}(t)^2] \leq \frac{c_5}{t \vee 1}.$$

Since

$$E[(t^2 D_\omega^{j,2}(t) - s^2 D_\omega^{j,2}(s))^2]$$
$$= \frac{1}{2}\int_s^t du_1 \int_0^{u_1} du_2 \int_s^t dv_1 \int_0^{v_1} dv_2 (\beta^j(u_2 - v_2) - \beta^j(u_2 + v_2))$$
$$\leq c_6(t - s)^2 t(1 \wedge t)$$

for $t > s$, we have

$$E[(D_\omega^{j,2}(t) - D_\omega^{j,2}(s))^2] \le \{2c_6|t-s|(t2 \wedge t) + 2|t^2 - s^2|^2 c_5/(s \vee 1)\}/t^4$$

and $\varphi(h) \le c_7 h/(1 \vee \sqrt{\eta R})$. Therefore we have

$$P\Big(\sup_{\eta R/2 \le t \le \eta R} |D_\omega^{j,2}(t)| \ge \frac{a_3}{\sqrt{\eta R} \vee 1}\Big) \le \frac{1}{3}$$

for some $0 < a_3 < \infty$. Similarly we have

$$P\Big(\sup_{R \le t \le R+1} |D_\omega^{j,2}(t)| \ge \frac{a_3}{\sqrt{R} \vee 1}\Big) \le \frac{1}{3}.$$

Therefore we have

$$P\Big(\sup_{\eta R/2 \le t \le \eta R \text{ or } R \le t \le R+1} |D_\omega^{j,2}(t)| \ge \frac{a_3}{\sqrt{\eta R} \vee 1}\Big) \ge \frac{1}{3}.$$

By the same argument, we have

$$P\Big(\sup_{\eta R/2 \le t \le \eta R \text{ or } R \le t \le R+1} |\overline{D_\omega^{j,1}}(t)| \ge \frac{a_4}{\sqrt{\eta R} \vee 1}\Big) \ge \frac{1}{3}$$

for some $0 < a_4 < \infty$. With these a_3 and a_4, we have

$$p_j(a_1, a_2) \ge \frac{1}{9} P\Big(\frac{\sqrt{R}}{c_3}\Big(a_1 + \frac{a_3 + a_4}{\sqrt{\eta R} \vee 1}\Big) \le W_\omega^j \le \frac{\sqrt{\eta R}}{c_4}\Big(a_2 - \frac{a_3 + a_4}{\sqrt{\eta R} \vee 1}\Big)\Big).$$

For any $a_1 > 0$, we take a_2 so that

$$\frac{\sqrt{\eta R}}{c_4}\Big(a_2 - \frac{a_3 + a_4}{\sqrt{\eta R} \vee 1}\Big) = \frac{\sqrt{R}}{c_3}\Big(a_1 + \frac{a_3 + a_4}{\sqrt{\eta R} \vee 1}\Big) + 1.$$

Then we have

$$a_2 = \frac{c_4}{c_3\sqrt{\eta}} a_1 + \Big(\frac{c_4}{c_3\sqrt{\eta}} + 1\Big)\frac{a_3 + a_4}{\sqrt{\eta R} \vee 1} + \frac{c_4}{\sqrt{\eta R}} \tag{6.3}$$

and

$$p_j(a_1, a_2) \ge \frac{1}{9\sqrt{2\pi}} \exp\Big(-\frac{\eta R}{2c_4^2}\Big(a_2 - \frac{a_3 + a_4}{\sqrt{\eta R} \vee 1}\Big)^2\Big) \ge c_8 \exp(-c_9 R a_1^2 - c_{10}/\eta).$$

We now assume that the events in the probability $p_j(a_1, a_2)$ with (6.3) for both $j=1$ and 2 occurs. Then we have

$$\frac{\|\mathcal{A}_\omega \phi^\omega_{n,R}\|^2}{\|\phi^\omega_{n,R}\|^2} \le \frac{c_{11}}{\eta R}\Big(\frac{6}{\eta}\Big)^{2n} \exp\Big(- R^2 \Big(a_1 - \frac{c_{12}\eta^{3/2}}{\sqrt{\eta R} \vee 1}\Big)\Big)$$

for $0 < \eta < c_{13}$ and $R \ge c_{14}$. Therefore, for $\|\mathcal{A}_\omega \phi^\omega_{n,R}\|^2 / \|\phi^\omega_{n,R}\|^2 \le \lambda$ to hold, it is sufficient that

$$n \le \frac{R^2(a_1 - c_{12}\eta^{3/2}/(\sqrt{\eta R} \vee 1)) + \log(\lambda \eta R / c_{11})}{2 \log(6/\eta)}.$$

Thus, by restricting to the above events, we have

$$\begin{aligned} N^-(\lambda) \ge & \frac{R^2(a_1 - c_{12}\eta^{3/2}/(\sqrt{\eta R} \vee 1)) + \log(\lambda \eta R / c_{11})}{2^2(R+1)^2 2 \log(6/\eta)} \\ & \times c_8^2 \exp(-2c_9 R a_1^2 - 2c_{10}/\eta). \end{aligned}$$

By taking $a_1 = 1/\sqrt{\eta R}$ and $R = (2\sqrt{\eta} \log(1/\lambda))^{2/3}$ so that

$$R^{3/2}/2 + \sqrt{\eta} \log \lambda = 0, \tag{6.4}$$

we have

$$N^-(\lambda) \ge c_{15} \frac{R^{3/2}(1/2 - c_{12}\eta^{3/2}) + \sqrt{\eta}\log(\eta R / c_{11})}{R^2} e^{-c_{16}/\eta}. \tag{6.5}$$

Since $R^{3/2}(1/2 - c_{12}\eta^{3/2}) + \sqrt{\eta}\log(\eta R/c_{11}) \ge R^{3/2}/3$ for $0 < \eta < c_{17}$ and $R \ge c_{18}$, we have

$$N^-(\lambda) \ge c_{19} e^{-c_{20}/\eta} \Big(\log \frac{1}{\lambda}\Big)^{-1/3}$$

for $0 < \lambda \le \exp(-c_{21}/\sqrt{\eta})$.

We next consider the case that the conditions (a-i), (a-ii) and (a-vii) are satisfied. We take the vector potential as

$$A^1_\omega = 0 \text{ and } A^2_\omega = \frac{d}{dx^1} C^1_\omega(x^1) = \int_0^{x^1} dt B^1_\omega(t).$$

For any R and $S > 0$, we restrict the Pauli Hamiltonian to the rectangle $(-R, R) \times (-S, S)$ by the Dirichlet and the periodic boundary conditions in the first and the second components, respectively: let $H_\omega^{-,R,[S]}$ be the self-adjoint operator on $L^2((-R, R) \times (-S, S))$ corresponding to the closure of the quadratic form $(\mathcal{A}_\omega \phi, \mathcal{A}_\omega \psi)$ with the domain

$$\{\phi \in C^1([-R, R] \times [-S, S]) : \phi(\pm R, \cdot) = 0, \phi(\cdot, S) = \phi(\cdot, -S)\}.$$

Now the functions $\phi_{n,S}^\omega(x) = \exp(-C_\omega^1(x^1) - 2\pi n z/S)$ satisfy the periodic condition in x^2 and $\mathcal{A}_\omega \phi_{n,S}^\omega = 0$ in the sense of distributions. Then the restricted functions $\phi_{n,R,S}^\omega(x) = \zeta_R(x^1)\phi_{n,S}^\omega(x)$ belong to the domain of the operator $H_\omega^{-,R+1,[S]}$ and satisfy

$$\|\mathcal{A}_\omega \phi_{n,R,S}^\omega\|^2 \le c_{22} S \exp\Big(-2R^2 \inf_{R \le |t| \le R+1} D_\omega^1(t) + 4\pi n(R+1)/S \Big)$$

when $\inf_{R \le |t| \le R+1} D_\omega^1(t) \ge 0$, and

$$\|\phi_{n,R,S}^\omega\|^2 \ge 2S\eta R \exp\Big(-2\eta^2 R^2 \Big(\sup_{\eta R/2 \le |t| \le \eta R} D_\omega^1(t) \Big)_+ - 4\pi n \eta R/S \Big).$$

When the event in the probability $p_1(a_1, a_2)$ with (6.3) occurs, we have

$$\frac{\|\mathcal{A}_\omega \phi_{n,R,S}^\omega\|^2}{\|\phi_{n,R,S}^\omega\|^2} \le \frac{c_{22}}{\eta R} \exp\Big(-R^2 \Big(a_1 - \frac{c_{23}\eta^{3/2}}{\sqrt{\eta R} \vee 1} \Big) + \frac{4\pi(R+1)n}{S} \Big)$$

for $0 < \eta < c_{24}$. Then, as in (6.5), the expectation of the counting function $N_\omega^-(\lambda : R, [S])$ of the eigenvalues of the operator $H_\omega^{-,R,[S]}$ is estimated as

$$E[N_\omega^-(\lambda : R+1, [S])] \ge c_{25} \frac{S}{R} \{R^{3/2}(1 - c_{23}\eta^{3/2}) + \sqrt{\eta} \log(\eta R/c_{22})\} e^{-c_{26}/\eta},$$

where R is taken as in (6.4). By the min-max principle, we see that

$$E[N_\omega^-(\lambda : N(R+1), [S])] \ge N E[N_\omega^-(\lambda : R+1, [S])]$$

for any $N \in \mathbb{N}$. Therefore, by taking the limit as $N \to \infty$ and using the uniqueness of the integrated density of states [19], [32], [45], we have

$$N^-(\lambda) \ge \frac{c_{27}}{R^2} \{R^{3/2}(1 - c_{23}\eta^{3/2}) + \sqrt{\eta} \log(\eta R/c_{22})\} e^{-c_{26}/\eta}$$

as in the preceding case.

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