Positivity of small ball probabilities of a Gaussian random field, and its applications to random Schrödinger operators

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Abstract. –

The spectrum of a Schrödinger operator with a Gaussian random potential is $\mathbb{R}$. This is a theorem appeared in a book by Pastur and Figotin and is proven by the theory on small ball probabilities. In this paper, the detail of the proof is discussed, and similar facts are proven for Dirac and Schrödinger operators with a Gaussian random magnetic field.

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1. Introduction

Let $(X^\omega(x))_{x\in\mathbb{R}^d}$ be a real Gaussian random field on $\mathbb{R}^d$ with $\mathbb{E}[X^\omega(x)] = 0$ and $\mathbb{E}[X^\omega(x)X^\omega(y)] = \gamma(x-y)$ satisfying

(A1) $\gamma(0) > 0$, $\lim\limits_{|x|\to 0}\gamma(x) = 0$, and γ is integrable and Hölder continuous at 0.

By the assumption (A1), we always realize the random field so that the sample path $X^\omega(x)$ is continuous in x (see Thereom 4.1.1 in Fernique [5]).

Our motivation is to prove the following:

Theorem 1. (i) *The spectral set* $\mathrm{spec}(-\Delta + (X^\omega)^2)$ *of the self-adjoint operator of the form* $-\Delta + (X^\omega)^2$ *on* $L^2(\mathbb{R}^d)$ *is* $[0,\infty)$*, where* $\Delta = \sum_{\iota=1}^d (\partial_\iota)^2$*.*

(ii) *The spectral set* $\mathrm{spec}(-\Delta + X^\omega)$ *of the self-adjoint operator of the form* $-\Delta + X^\omega$ *on* $L^2(\mathbb{R}^d)$ *is* $\mathbb{R}$*.*

The results of this theorem are stated in Theorem II-(5.34) in Pastur and Figotin [17] with the outline of the proof. In this paper we give a detailed proof of this theorem and extend the results to other operators: Dirac and Schrödinger operators with a Gaussian random magnetic field.

The key theorem for the proof of Theorem 1 is the following:

Theorem 2. (i) *It holds that*

$$\mathbb{P}(\sup_{|x|\le\ell}|X^\omega(x)|<\eta)>0 \tag{1.1}$$

for any $\ell,\eta>0$.

(ii) *We assume*

(A2) $\int_{|\zeta|<\delta}\widehat{\gamma}(\zeta)d\zeta>0$ *for any* $\delta>0$, *where*

$$\widehat{f}(\zeta)=\int_{\mathbb{R}^d}\exp(-2\pi i\zeta\cdot x)f(x)dx$$

is the Fourier transform.

Then it holds that

$$\mathbb{P}(\sup_{|x|\le\ell}|X^\omega(x)-\lambda|<\eta)>0 \tag{1.2}$$

for any $\ell,\eta>0$ *and* $\lambda\in\mathbb{R}$.

(iii) *There exists a non-zero real rapidly decreasing function* V *on* $\mathbb{R}^d$ *such that*

$$\mathbb{P}(\sup_{|x|\le\ell}|X^\omega(x)-\lambda V(x)|<\eta)>0 \tag{1.3}$$

for any $\ell,\eta>0$ *and* $\lambda\in\mathbb{R}$.

The probability in this theorem is called the small ball probability and is well studied. For this aspect, see Li and Shao [14]. Theorem 2 (i) is a simple consequence from more detailed results by Talagrand [20] and Ledoux [12] Section 7 (see also Li and Shao [14] Theorem 3.8). By Theorem 2 (i), we can show

$$\mathrm{spec}(-\Delta+(X^\omega)^2),\mathrm{spec}(-\Delta+X^\omega)\supset\mathrm{spec}(-\Delta)=[0,\infty) \tag{1.4}$$

and the proof of Theorem 1 (i) is completed. Theorem 2 (ii) and (iii) are simple consequences from more detailed results obtained by Hoffmann-Jorgensen, Shepp and Dudley [8] and de Acosta [4] on the small ball probability in the case that the ball is shifted (see also Li and Shao [14] Theorem 3.1, Theorem 3.2). Their tool is the Cameron and Martin theorem [3]. In [17], the estimate (1.2) is stated without proof and is used to prove Theorem 1. In this paper, we show that the estimate (1.2) is proven by the Cameron and

Martin theorem under the assumption (A2), and we use Theorem 2 (iii) to prove Theorem 1 (ii) so that the assumption (A2) is not required. For Dirac and Schrödinger operators with random magnetic fields, Theorem 2 (iii) is difficult to apply and we apply Theorem 2 (ii) for the identification of the spectral sets.

To obtain Theorem 1 from Theorem 2, we deduce the following as in Ando, Iwatsuka, Kaminaga and Nakano [1]:

Lemma 1.1. (i) *For almost all* ω*, any* $\ell, k \in \mathbb{N}$*, there exists* $x(\omega, \ell, k) \in \mathbb{R}^d$ *such that* $\sup_{|x-x(\omega,\ell,k)|\le \ell} |X^\omega(x)| < 1/k$.

(ii) *Under (A2), for almost all* ω*, any* $\ell, k \in \mathbb{N}$ *and* $\lambda \in \mathbb{R}$*, there exists* $x(\omega, \lambda, \ell, k) \in \mathbb{R}^d$ *such that* $\sup_{|x-x(\omega,\lambda,\ell,k)|\le \ell} |X^\omega(x) - \lambda| < 1/k$.

(iii) *For almost all* ω*, any* $\ell, k \in \mathbb{N}$ *and* $\lambda \in \mathbb{R}$*, there exists* $x(\omega, \lambda V, \ell, k) \in \mathbb{R}^d$ *such that* $\sup_{|x|\le \ell} |X^\omega(x + x(\omega, \lambda V, \ell, k)) - \lambda V(x)| < 1/k$*, where* V *is the function given in Theorem 1* (iii).

For comprehensive reviews on the random Scrödinger operators, refer to Carmona and Lacroix [2] and Pastur and Figotin [17]. For the spectrum of the operator $-\Delta + X^\omega$, the asymptotic distribution at low energies is studied by Pastur [16] and the Anderson localization at low energies is proven by Fischer, Leschke, and Müller [6]. For the spectrum of the operator $-\Delta + (X^\omega)^2$, the asymptotic distribution at low energies is studied in [21].

The organization of this paper is as follows. In Section 2, we discuss on the proof of Theorem 2. In Section 3, we discuss on the proof of Theorem 1. In Section 4, we extend the identification of the spectral sets to Dirac and Scrödinger operators with a Gaussian random magnetic field.

2. Proof of Theorem 2

Proof of Theorem 2 (i).

We have a positive lower bound

$$\mathbb{P}(\sup_{|x|\le \ell} |X^\omega(x)| < \eta) \ge \exp(-\psi(\eta)), \tag{2.1}$$

under the existence of a nonnegative function $\psi(\eta)$ satsfying

$$N(\{|x| \le \ell\}, \gamma, \eta) \le \psi(\eta)$$

and

$$C_1\psi(\eta) \le \psi\left(\frac{\eta}{2}\right) \le C_2\psi(\eta),$$

with some constants $C_1, C_2 \in (0,\infty)$, where $N(\{|x| \le \ell\}, \gamma, \eta)$ is the entropy number defined by

$$\min\{n \in \mathbb{N} : \text{ there exist } \{x_i\}_{i=1}^n \subset \mathbb{R}^d \text{ s.t. } \bigcup_{i=1}^n B_\gamma(x_i,\eta) \supset \{|x| \le \ell\}\},$$

$B_\gamma(x_i,\eta) = \{x \in \mathbb{R}^d : d_\gamma(x,x_i) < \eta\}$, and $d_\gamma(\cdot,\cdot)$ is the Dudley metric defined by

$$d_\gamma(x,y) = \mathbb{E}[(X^\omega(x) - X^\omega(y))^2]^{1/2}.$$

The bound in (2.1) was obtained by Talagrand [20] and the formulation in (2.1) was given by Ledoux [12] Section 7 (see also Li and Shao [14] Theorem 3.8). We can take $\psi(\eta) = c\ell^d/\eta^{d/\alpha}$ if γ is α-Hölder continuous at 0 in the assumption (A1), where c is some positive finite constant. □

For the bound in (2.1), $\mathbb{E}[X^\omega(x)] = 0$ is essential since this is also essential for the main tool for the proof of (2.1), which is the correlation inequality

$$\mathbb{P}(\max_{1\le i\le n} |X^\omega(x_i)| < \eta) \ge \mathbb{P}(|X^\omega(x_1)| < \eta)\mathbb{P}(\max_{2\le i\le n} |X^\omega(x_i)| < \eta),$$

This correlation inequality was proven by Khatri [11] and Sidak [18], [19] (See also Section 2.2 in Li and Shao [14]).

Proof of Theorem 2 (ii).

To treat shifted small balls generally, the Cameron and Martin theorem [3] is applied with Hölder's inequality as

$$\mathbb{P}(\sup_{|x|\le\ell} |X^\omega(x) - h(x)| < \eta) \ge \mathbb{P}(\sup_{|x|\le\ell} |X^\omega(x)| < \eta)\exp(-\|h\|_H^2/2),$$

where $\|\cdot\|_H$ is the norm of the Cameron and Martin space (see Hoffmann-Jorgensen, Shepp and Dudley [8], de Acosta [4], Section 3.1 in Li and Shao [14]). In our case, the norm is

$$\|h\|_H = \Big(\int_{\mathbb{R}^d} \frac{|\widehat{h}(\zeta)|^2}{\widehat{\gamma}(\zeta)} d\zeta\Big)^{1/2}.$$

Therefore a sufficient condition for the result of Theorem 2 (ii) is the following:

(A3) For any $\ell, \eta > 0$, there exists a function h on $\mathbb{R}^d$ such that

$$\sup_{|x|\le\ell} |1 - h(x)| < \eta \tag{2.2}$$

and

$$\int_{\mathbb{R}^d} \frac{|\widehat{h}(\zeta)|^2}{\widehat{\gamma}(\zeta)} d\zeta < \infty. \tag{2.3}$$

We will show that (A3) holds under (A2). For any $\delta > 0$, there exists $\varepsilon > 0$ such that $|\{|\zeta| < \delta : \widehat{\gamma}(\zeta) > \varepsilon\}| > 0$, where $|\cdot\|$ is the d-dimensional Lebesgue measure. Since $\{|\zeta| < \delta : \widehat{\gamma}(\zeta) > \varepsilon\}$ is nonempty open set, we can take a nonnegative continuous function g such that

$$\operatorname{supp} g \subset \{|\zeta| < \delta : \widehat{\gamma}(\zeta) > \varepsilon\} \tag{2.4}$$

and

$$\int g(\zeta)\sqrt{\widehat{\gamma}(\zeta)}d\zeta = 1. \tag{2.5}$$

Since $\widehat{\gamma}(-\zeta) = \widehat{\gamma}(\zeta)$, we can take this function g so that $g(-\zeta) = g(\zeta)$. Then the function

$$h(x) = \int \exp(2\pi\zeta \cdot x) g(\zeta)\sqrt{\widehat{\gamma}(\zeta)}d\zeta$$

is real valued, satisfies (2.3) by (2.4), and satisfies (2.2) by (2.4) and (2.5) if $\delta < \eta/(2\pi\ell)$. □

Proof of Theorem 2 (iii).

We can take a function V so that $\widehat{V}$ is a non-zero smooth function such that $\widehat{V}(-\zeta) = \widehat{V}(\zeta)$ and $\operatorname{supp}\widehat{V}$ is a compact set included in $\{\zeta : \widehat{\gamma}(\zeta) > \varepsilon\}$ with sufficiently small $\varepsilon > 0$. □

3. Proof of Theorem 1

We first deduce Lemma 1.1 from Theorem 2 as in Ando, Iwatsuka, Kaminaga and Nakano [1]:

Proof of Lemma 1.1.

We prove only (ii). Other statements are similarly proven. For any $\lambda \in \mathbb{R}$ and $k, \ell \in \mathbb{N}$, we define the events

$$E(\lambda, \ell, k) := \Big\{\omega : \sup_{|x|\le\ell} |X^\omega(x) - \lambda| < \frac{1}{k}\Big\}$$

and

$$D(\lambda, \ell, k) := \Big\{\omega : \sup_{|x-x_0|\le\ell} |X^\omega(x) - \lambda| < \frac{1}{k} \text{ for some } x_0 \in \mathbb{R}^d\Big\}.$$

Then by Theorem 2,

$$\mathbb{P}(E(\lambda, \ell, k)) > 0,$$

$$E(\lambda, \ell, k) \subset D(\lambda, \ell, k),$$

and $D(\lambda, \ell, k)$ is invariant under the shift in the space variable. Since the random field $\{X^\omega(x)\}$ is metrically transitive, we have

$$\mathbb{P}(D(\lambda, \ell, k)) = 1$$

and

$$\mathbb{P}\Big(\bigcap_{k,\ell\in\mathbb{N}} D(\lambda,\ell,k)\Big)=1.$$

The metrically transitivity of the random fields are discussed in a more general framework of stationary fields in Maruyama [15]. Thus for $\omega\in\bigcap_{k,\ell\in\mathbb{N}} D(\lambda,\ell,k)$ and $k,\ell\in\mathbb{N}$, there exists $x(\omega,\lambda,\ell,k)\in\mathbb{R}^d$ such that

$$\sup_{|x-x(\omega,\lambda,\ell,k)|\le\ell}|X^\omega(x)-\lambda|<\frac{1}{k}.$$

□

We next prove Theorem 1, referring Pastur and Figotin [17].

Proof of Theorem 1.

We first prove (1.4). For any $\mu\in[0,\infty)$, there exist $\{\varphi_n\}_{n\in\mathbb{N}}$ in $C_0^\infty(\mathbb{R}^d)$ such that $\|\varphi_n\|_{L^2(\mathbb{R}^d)}=1$ and $\|(-\Delta-\mu)\varphi_n\|_{L^2(\mathbb{R}^d)}\to 0$ as $n\to\infty$ by Weyl's criterion (see Hislop and Sigal [7], Theorem 5.10). Let $\ell(n)$ be an integer such that $\operatorname{supp}\varphi_n\subset\{|x|<\ell(n)\}$, and let $x(\omega,\ell(n),n)$ be the point in $\mathbb{R}^d$ given in Lemma 1.1 (i). Then, for any $m\in\mathbb{N}$, we have

$$\begin{aligned}&\|(-\Delta+(X^\omega)^m-\mu)\varphi_n(\cdot-x(\omega,\ell(n),n))\|_{L^2(\mathbb{R}^d)}\\ \le&\|(-\Delta-\mu)\varphi_n\|_{L^2(\mathbb{R}^2)}+\sup_{|x-x(\omega,\ell(n),n)|\le\ell(n)}|X^\omega(x)|^m,\end{aligned}$$

which converges to 0 as $n\to\infty$. Thus by Weyl's criterion, we have $\mu\in\operatorname{spec}(-\Delta+(X^\omega)^m)$.

We next show that $(-\infty,0)\subset\operatorname{spec}(-\Delta+X^\omega)$ by applying Theorem 2 (iii). For any $\mu\in(-\infty,0)$, there exists $\lambda_\mu\in\mathbb{R}$ such that

$$\inf\operatorname{spec}(-\Delta+\lambda_\mu V)=\mu,$$

where V is the function given in Theorem 2 (iii). Indeed, by the expression

$$\inf\operatorname{spec}(-\Delta+\lambda V)=\inf\Big\{\int_{\mathbb{R}^d}dx(|\nabla\varphi|^2*\lambda V(x)|\varphi(x)|^2):\varphi\in C_0^\infty(\mathbb{R}^d),\|\varphi\|_{L^2(\mathbb{R}^d)}=1\Big\},$$

we see that $\inf\operatorname{spec}(-\Delta+\lambda V)$ is a continuous function of λ such that $\inf\operatorname{spec}(-\Delta)=0$ at $\lambda=0$. If $V\le-\varepsilon$ on a non-empty open set S with a positive number ε, then we have

$$\inf\operatorname{spec}(-\Delta+\lambda V)\le\inf\operatorname{spec}(-\Delta_S)-\lambda\epsilon$$

for $\lambda>0$, where Δ_S is the Laplacian on S with the Dirichlet boundary condition. Since the right hand side tends to $-\infty$ as $\lambda\uparrow\infty$, we see the existence of λ_μ in $(0,\infty)$. If $V\ge\varepsilon$ on a non-empty open set S, then we see the existence of λ_μ in $(-\infty,0)$.

There exist $\{\varphi_n\}_{n\in\mathbb{N}}$ in $C_0^\infty(\mathbb{R}^d)$ such that $\|\varphi_n\|_{L^2(\mathbb{R}^d)} = 1$ and $\|(-\Delta + \lambda_\mu V - \mu)\varphi_n\|_{L^2(\mathbb{R}^d)} \to 0$ as $n \to \infty$ by Weyl's criterion. Let $\ell(n)$ be an integer such that $\operatorname{supp}\varphi_n \subset \{|x| < \ell(n)\}$, and let $x(\omega, \lambda_\mu V, \ell(n), n)$ be the point in $\mathbb{R}^d$ given in Lemma 1.1 (iii). Then

$$\begin{aligned}&\|(-\Delta + X^\omega(x) - \mu)\varphi_n(x - x(\omega, \lambda_\mu V, \ell(n), n))\|_{L^2(\mathbb{R}^d)}\\ \le&\|(-\Delta + \lambda_\mu V - \mu)\varphi_n\|_{L^2(\mathbb{R}^2)} + \sup_{|x|\le\ell(n)} |X^\omega(x + x(\omega, \lambda_\mu V, \ell(n), n)) - \lambda_\mu V(x)|,\end{aligned}$$

which converges to 0 as $n \to \infty$. Thus by Weyl's criterion, we have $\mu \in \operatorname{spec}(-\Delta + X^\omega)$. □

4. Applications to operators for magnetic fields

In this section, we consider operators on $\mathbb{R}^2$. We first consider the Dirac operator

$$D^\omega := \sum_{\iota=1}^{2} \gamma_\iota(i\partial_\iota + A_\iota^\omega(x)), \tag{4.1}$$

where

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \gamma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

are Pauri matrices. $A^\omega(x) = A^0(x) + \widetilde{A^\omega}(x)$, $A^0(x)$ is a vector potential of a uniform magnetic field $B \in [0, \infty)$, and $\widetilde{A^\omega}(x)$ is a random vector potential of a random magnetic field $\widetilde{B^\omega}(x)$, which is a Gaussian random field on $\mathbb{R}^2$ with $\mathbb{E}[\widetilde{B^\omega}(x)] = 0$ and $\mathbb{E}[\widetilde{B^\omega}(x)\widetilde{B^\omega}(y)] = \gamma(x - y)$ satisfying (A1). For example, we can take the potentials as

$$A^0(x) = \frac{B}{2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

and

$$A^\omega(x) = \int_0^1 dt\ t\widetilde{B^\omega}(tx)\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}. \tag{4.2}$$

The operator D^ω in (4.1) with the domain $C_0^\infty(\mathbb{R}^2 \to \mathbb{C}^2)$ is essentially self-adjoint in the Hilbert space $L^2(\mathbb{R}^2 \to \mathbb{C}^2)$, where $C_0^\infty(\mathbb{R}^2 \to \mathbb{C}^2)$ is the set of all $\mathbb{C}^2$-valued smooth functions with compact supports and $L^2(\mathbb{R}^2 \to \mathbb{C}^2)$ is the Hilbert space of all $\mathbb{C}^2$-valued L^2-functions (see Jörgen [10], Ivrii [9] §9.2.1). We denote the unique self-adjoint extension by the same symbol. For this operator, we will show the following:

Proposition 4.1. (i) *Under (A1) and either (A2) or* $B = 0$*, the spectral set* $\mathrm{spec}(D^\omega)$ *of* D^ω *is* $\mathbb{R}$*.*

(ii) *Under (A1) and* $B \neq 0$*, we have*

$$\mathrm{spec}(D^\omega) \supset \{\pm\sqrt{2nB} : n \in \mathbb{Z}_+\}.$$

The right hand side is the Landau level of the Dirac operator D^0 *for the uniform magnetic field* B *and* $\widetilde{B^\omega} = 0$*.*

Proof.

(i) For any $\mu \in \mathbb{R}$, there exist $\{\varphi_n\}_{n\in\mathbb{N}}$ in $C_0^\infty(\mathbb{R}^2 \to \mathbb{C}^2)$ such that $\|\varphi_n\|_{L^2(\mathbb{R}^2)} = 1$ and $\|(D_0 - \mu)\varphi_n\|_{L^2(\mathbb{R}^2)} \to 0$ as $n \to \infty$ by Weyl's criterion (Hislop and Sigal [7], Theorem 5.10), where

$$D_0 = \sum_{\iota=1}^{2} \gamma_\iota i\partial_\iota.$$

For almost all ω and any $\ell, k \in \mathbb{N}$, we take $x(\omega, B, \ell, k) \in \mathbb{R}^2$ such that

$$\sup_{|x - x(\omega,B,\ell,k)| \le \ell} |\widetilde{B^\omega}(x) + B| < 1/k \tag{4.3}$$

by Lemma 1.1. For any $n \in \mathbb{N}$, there exists $\ell(n) \in \mathbb{N}$ such that

$$\mathrm{supp}\,\varphi_n \subset \{x : |x| \le \ell(n)\}.$$

Then we have

$$|\widetilde{B^\omega}(x) + B| < 1/k$$

on $\{x \in \mathbb{R}^2 : |x - x(\omega, B, \ell(n), k)| \le \ell(n)\} \supset \mathrm{supp}\,\varphi_n(\cdot - x(\omega, B, \ell(n), k))$. On this set, we take the vector potential as

$$A^\omega(x) = \int_0^1 dt\; t(B + \widetilde{B^\omega}(tx + (1-t)x(\omega, B, \ell(n), k))) \begin{pmatrix} -x_2 + x_2(\omega, B, \ell(n), k) \\ x_1 - x_1(\omega, B, \ell(n), k) \end{pmatrix}, \tag{4.4}$$

which satisfies

$$|A^\omega(x)| \le \frac{\ell(n)}{2k}.$$

Thus we have

$$\begin{aligned} &\|(D^\omega - \mu)\varphi_n(\cdot - x(\omega, B, \ell(n), k))\|_{L^2(\mathbb{R}^2)} \\ \le &\|(D_0 - \mu)\varphi_n\|_{L^2(\mathbb{R}^2)} + \ell(n)/k. \end{aligned}$$

Thus by Weyl's criterion, μ belongs to the spectral set of D^ω.

(ii) In the proof of (i), let μ be an element of $\mathrm{spec}(D^0)$. For almost all ω and any $\ell, k \in \mathbb{N}$, we take $x(\omega, \ell, k) \in \mathbb{R}^2$ such that

$$\sup_{|x - x(\omega,\ell,k)| \leq \ell} |\widetilde{B^\omega}(x)| < 1/k \tag{4.5}$$

by Proposition 1.1, and we take $\ell(n) \in \mathbb{N}$ such that $\mathrm{supp}\, \varphi_n \subset \{x : |x| \leq \ell(n)\}$. We divide the vector potential similar to that in (4.4) as $A^\omega(x) = A^\omega(x)^1 + A^\omega(x)^2$, where

$$A^\omega(x)^1 = \frac{B}{2} \begin{pmatrix} -x_2 + x_2(\omega, B, \ell(n), k) \\ x_1 - x_1(\omega, B, \ell(n), k) \end{pmatrix}, \tag{4.6}$$

and

$$A^\omega(x)^2 = \int_0^1 dt\ t\widetilde{B^\omega}(tx + (1-t)x(\omega, \ell(n), k)) \begin{pmatrix} -x_2 + x_2(\omega, \ell(n), k) \\ x_1 - x_1(\omega, \ell(n), k) \end{pmatrix}. \tag{4.7}$$

On the set $\{x \in \mathbb{R}^2 : |x - x(\omega, \ell(n), k)| \leq \ell(n)\}$, we have

$$|A^\omega(x)^2| \leq \frac{\ell(n)}{2k}.$$

Then we have

$$\begin{aligned} &\|(D^\omega - \mu)\varphi_n(\cdot - x(\omega, \ell(n), k))\|_{L^2(\mathbb{R}^2)} \\ \leq &\|(D^0 - \mu)\varphi_n\|_{L^2(\mathbb{R}^2)} + \ell(n)/k. \end{aligned}$$

Thus by Weyl's criterion, μ belongs to the spectral set of D^ω. □

We next consider the Schrödinger operator

$$H^\omega := \sum_{\iota=1}^{2} (i\partial_\iota + A_\iota^\omega(x))^2. \tag{4.8}$$

We define this operator as the unique self-adjoint operator on $L^2(\mathbb{R}^2 \to \mathbb{C})$ associated with the closure of the quadratic form

$$\sum_{\iota=1}^{2} \int dx (i\partial_\iota \varphi(x) + A_\iota^\omega(x)\varphi(x))\overline{(i\partial_\iota \psi(x) + A_\iota^\omega(x)\psi(x))}$$

with the domain $C_0^\infty(\mathbb{R}^2 \to \mathbb{C})$, where $C_0^\infty(\mathbb{R}^2 \to \mathbb{C})$ is the set of all $\mathbb{C}$-valued smooth functions with compact supports and $L^2(\mathbb{R}^2 \to \mathbb{C})$ is the Hilbert space of all $\mathbb{C}$-valued L^2-functions. The domain of the closed form coincides with

$$\{\varphi \in L^2(\mathbb{R}^2 \to \mathbb{C}) : i\partial_\iota \varphi + A_\iota^\omega \varphi \in L^2(\mathbb{R}^2 \to \mathbb{C}) \text{ for } \iota \in \{1, 2\}\}.$$

If we can take A^ω so that $\nabla \cdot A^\omega$ is L^2 on any compact sets, then this operator is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2 \to \mathbb{C})$ (see Leinfelder and Simader [13]). For this operator, we will show the following:

Proposition 4.2. (i) *Under (A1) and either (A2) or* $B = 0$*, the spectral set* $\mathrm{spec}(H^\omega)$ *of* H^ω *is the interval* $[0, \infty)$.

(ii) *Under (A1) and* $B \neq 0$*, we have*

$$\mathrm{spec}(H^\omega) \supset \{(2n+1)B : n \in \mathbb{Z}_+\}.$$

The right hand side is the Landau level of the Landau Hamiltonian H^0 *for the uniform magnetic field* B *and* $\widetilde{B^\omega} = 0$.

Proof.

(i) For any $\mu \in [0, \infty)$, there exist $\{\varphi_n\}_{n \in \mathbb{N}}$ in $C_0^\infty(\mathbb{R}^2)$ such that $\|\varphi_n\|_{L^2(\mathbb{R}^2)} = 1$ and $\|(-\Delta - \mu)\varphi_n\|_{L^2(\mathbb{R}^2)} \to 0$ as $n \to \infty$ by Weyl's criterion (Hislop and Sigal [7], Theorem 5.10). We take $x(\omega, B, \ell, k) \in \mathbb{R}^d$ as in (4.3). However the vector potential in (4.4) needs to be modified, since $\nabla \cdot A^\omega$ appears on the operator H^ω. Thus we use a vector potential of the Coulomb gauge: we take $\ell(n) \in \mathbb{N}$ so that

$$\mathrm{supp}\, \varphi_n \subset \{x : |x| \leq \ell(n) - 1\},$$

and take a vector potential as

$$A^\omega(x) = \int_{\mathbb{R}^2} dy \widetilde{\chi_{\ell(n)}}(y - x(\omega, B, \ell, k)) \frac{B + \widetilde{B^\omega}(y)}{2\pi|x - y|^2} \begin{pmatrix} -x_2 + y_2 \\ x_1 - y_1 \end{pmatrix}, \tag{4.9}$$

where $\widetilde{\chi_{\ell(n)}}$ is a $[0, 1]$-valued smooth function on $\mathbb{R}^2$ such that $\widetilde{\chi_{\ell(n)}}(x) = 1$ for $|x| \leq \ell(n) - 1$ and $\widetilde{\chi_{\ell(n)}}(x) = 0$ for $|x| \geq \ell(n)$. Then we have $\nabla \cdot A^\omega(x) = 0$ and

$$|A^\omega(x)| \leq 2\ell(n)/k$$

on $\mathrm{supp}\, \varphi_n(\cdot - x(\omega, B, \ell(n), k)) \subset \{x : |x - x(\omega, B, \ell(n), k)| \leq \ell(n) - 1\}$. Thus $\varphi_n(\cdot - x(\omega, B, \ell(n), k))$ belongs to the domain of H^ω,

$$H^\omega \varphi_n(\cdot - x(\omega, B, \ell(n), k)) = (-\Delta + 2iA^\omega \cdot \nabla + |A^\omega|^2)\varphi_n(\cdot - x(\omega, B, \ell(n), k)),$$

and we can estimate as

$$\begin{aligned} &\|(H^\omega - \mu)\varphi_n(\cdot - x(\omega, B, \ell(n), k))\|_{L^2(\mathbb{R}^2)} \\ \leq &\|(-\Delta - \mu)\varphi_n\|_{L^2(\mathbb{R}^2)} + \frac{4\ell(n)}{k}\|\nabla\varphi_n\|_{L^2(\mathbb{R}^2)} + \Big(\frac{2\ell(n)}{k}\Big)^2 \end{aligned}$$

and

$$\|\nabla\varphi_n\|_{L^2(\mathbb{R}^2)} \leq \sqrt{\|(-\Delta-\mu)\varphi_n\|_{L^2(\mathbb{R}^2)} + \mu}.$$

By Weyl's criterion, we see that μ belongs to the spectral set of H^ω.

(ii) In the proof of (i), let μ be an element of $\mathrm{spec}(H^0)$. For almost all ω and any $\ell, k \in \mathbb{N}$, we take $x(\omega, \ell, k) \in \mathbb{R}^2$ as in (4.5), and we take $\ell(n) \in \mathbb{N}$ such that $\operatorname{supp}\varphi_n \subset \{x : |x| \leq \ell(n) - 1\}$. We divide the vector potential similar to that in (4.9) as $A^\omega(x) = A^\omega(x)^1 + A^\omega(x)^2$, where $A^\omega(x)^1$ is same with (4.6), and

$$A^\omega(x)^2 = \int_{\mathbb{R}^2} dy \widetilde{\chi_{\ell(n)}}(y - x(\omega, B, \ell, k)) \frac{\widetilde{B^\omega}(y)}{2\pi|x-y|^2} \begin{pmatrix} -x_2 + y_2 \\ x_1 - y_1 \end{pmatrix}.$$

On the set $\{x \in \mathbb{R}^2 : |x - x(\omega, \ell(n), k)| \leq \ell(n) - 1\}$, we have

$$|A^\omega(x)^2| \leq \frac{2\ell(n)}{k}$$

and

$$|A^\omega(x)^2| \leq \frac{B\ell(n)}{2}.$$

Then we have

$$\begin{aligned} &\|(H^\omega - \mu)\varphi_n(\cdot - x(\omega, \ell(n), k))\|_{L^2(\mathbb{R}^2)} \\ \leq &\|(H^0 - \mu)\varphi_n\|_{L^2(\mathbb{R}^2)} + \frac{4\ell(n)}{k}\|\nabla\varphi_n\|_{L^2(\mathbb{R}^2)} + \ell(n)^2 \frac{2}{k}\Big(\frac{2}{k} + B\Big) \end{aligned}$$

and

$$\|\nabla\varphi_n\|_{L^2(\mathbb{R}^2)} \leq \sqrt{\|(H^0-\mu)\varphi_n\|_{L^2(\mathbb{R}^2)} + \mu}.$$

By Weyl's criterion, we see that μ belongs to the spectral set of H^ω.

□

For the spectrum of the operator H^ω, the asymptotic distribution and the Anderson localization at low energies are studied in [21] and [22].

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